Calculus for ELECTRONICS

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III

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Preface

This book is one of applications of mathematics. I have intended to present those methods and results from the calculus which are of the most direct use in the study of circuits. In order to use this book, the reader should previously have acquainted himself with the fundamentals of algebra and trigonometry, including operations with logarithms and complex numbers. A working familiarity with electric and electronic circuits is desirable, but this does not have to be extensive.

The emphasis upon utility in this book is illustrated by these features:

1. There are included such topics as the Taylor's-series representation of functions of two independent variables, a chapter on the Fourier series, the complex-exponential representation of physical sinusoids (not often seen in textbooks), and an introductory chapter on differential equations.
2. The rationalized mks system of units is used almost exclusively.
3. Certain traditional topics are omitted—such as moments of inertia, centroids, and radii of gyration. These topics, however interesting and useful, are for present purposes best replaced by treatments of differentiating and integrating circuits, Kirchhoff's laws, electric transients, and the Weber-Fechner law.
4. The symbols used were chosen with regard to published standards, particularly ASA Z10f-1928, "American Standard Mathematical Symbols," ASA Z10.1-1941, "American Standard Abbreviations for Scientific and Engineering Terms," and 51 IRE 21.S1, "Standards on Abbreviations of Radio-Electronic Terms." In many instances, I have given alternative terms or symbols to aid the reader in his work outside this book.

There are a great many problems relating directly to the art of electronics—approximately four hundred and fifty such problems, in fact. In addition, there are general "x and y" problems, which should be useful as drill exercises. Answers for most odd-numbered problems are given in the back of the book.

It is my hope that this book will prove useful as a mathematics text in technical institutes and in training-in-industry programs and as an introductory text in conjunction with courses in circuit analysis in colleges and universities and in the armed services. Practicing engineers might find this work of use in reviewing calculus procedures.

It is a pleasure to acknowledge the cooperation and help which I have had in the preparation of this book. This cooperation has come from sources so numerous that they cannot be individually mentioned.

I should be grateful if users would inform me of errors which they may find in this book or of their suggestions for its improvement. This information may be sent in care of the publishers.

A. E. Richmond

Contents

Part One

FUNDAMENTAL CONCEPTS

Greek Alphabet

Upper-case	Lower-case	Name	English equivalent
Α	α	Alpha	a
Β	β	Beta	b
Γ	γ	Gamma	g
Δ	δ	Delta	d
Ε	ϵ	Epsilon	ĕ
Ζ	ζ	Zeta	z
Η	η	Eta	ē
Θ	θ	Theta	th
Ι	ι	Iota	i
Κ	κ	Kappa	k
Λ	λ	Lambda	l
Μ	μ	Mu	m
Ν	ν	Nu	n
Ξ	ξ	Xi	x
Ο	o	Omicron	ŏ
Π	π	Pi	p
Ρ	ρ	Rho	r
Σ	σ, ς	Sigma	s
Τ	τ	Tau	t
Υ	υ	Upsilon	u
Φ	ϕ, φ	Phi	ph, f
Χ	χ	Chi	ch
Ψ	ψ	Psi	ps
Ω	ω	Omega	ō

1

Introduction

Electronics technicians and engineers have learned how to solve a variety of equations in order to work the problems encountered in their work. For example, the power in a resistor at any instant is known to be

$$p = i^2R$$

where i is the instantaneous current and R is the resistance of the resistor. This equation can readily be rearranged so that the value of p, i, or R can be found if the other two are given. In calculus we perform algebraic operations often no more difficult than those in the algebra problems just mentioned, but in calculus the quantities which we solve for are, in general, different from those found in algebra problems.

1-1 What is calculus? In the above equation, we were interested perhaps in finding the *amount* of power corresponding to a certain value of current in the resistor; while in calculus we may be more concerned with the *rate* at which the power *changes* as the current in the resistor is varied. In particular, we may want to know the rate of change of power *at a given instant.* The idea of an *instantaneous* rate of change of one quantity, as an associated quantity changes, is a basic one in calculus. You will recall that no methods of handling such problems were presented

in algebra; only constant rates and average rates were considered. Ordinary algebraic methods are used in calculus, but we shall first develop a few ideas in order to know how to apply them.

The importance of knowing the rate of change of a quantity is clear. A familiar example is to be found in the transformer. The voltage induced in the secondary circuit is proportional to the *rate of change* of the current in the primary, so that, if we can find by calculus the exact rate of change of current in the primary winding at any instant, we can then figure the induced secondary voltage at that instant.

A second major problem dealt with in calculus is this: given the equation for the rate of change of a quantity, *what is the equation for the quantity itself?* We know, for instance, that the value of the current in a circuit is actually an expression of the *rate* at which electric charges are being moved, one ampere of current representing a rate of change of one coulomb per second. Suppose that a charging current is supplied to a capacitor and that the current changes with time according to a known equation. Then, by means of calculus, it is possible to compute the total charge delivered to the capacitor within a certain interval (and therefore, to determine the resulting voltage across its terminals). You can see that this type of problem is actually the inverse of the first kind, where the rate of change was determined from the equation for the quantity itself.

It can be said, then, that calculus is a branch of mathematics which deals with the two large classes of problems mentioned above. You will be able to solve many other kinds of problems as a direct result of learning the methods for these two kinds.

1-2 Why learn calculus? The problem of the rates of change of quantities exists universally. In electrical work, as in nearly all fields of science or practice, we are seriously handicapped if we are able to figure only the amounts of the quantities we work with. We have noted, for instance, that the *rate of change* of one quantity may have to be found in order to determine the *amount* of another quantity.

Likewise, we often have an equation for the rate of change of a quantity and need to find the value of that quantity.

Problems of these kinds may be solved, in general, *only* by calculus methods. It is unfortunately true that during your previous training you probably have not been informed of the many kinds of problems you have *not* been trained to solve. Only by mastering the subject, then, are you likely to develop an appreciation of the power contained in calculus methods.

Not the least value in a calculus course consists in the methods of thinking which it teaches. Even if you forget the precise methods of working the problems, you, as a calculus student, can carry with you a

knowledge of the processes of reasoning and of nature which go far beyond the limitations of people who have never been trained in calculus. Calculus is, then, first of all *a method of thinking*, and second, a method of working problems.

1-3 Is calculus difficult? Calculus is an orderly and reasonable course. It has much in common with other courses in mathematics. For example, consider the materials with which we shall work:

1. Information obtained in previous courses, such as algebra.
2. Material which is so simple as to be self-evident (but which will nevertheless be fully explained).

By short steps in reasoning we combine these materials to produce results which are more useful than were the original materials alone. This is the time-honored method of learning mathematics. There is nothing difficult about calculus that is not also difficult about other courses which follow this procedure. In this way you have learned what you now know concerning mathematics.

The title *calculus* has unfortunately become the object of unnecessary dread on the part of some students. It is true that the course is more advanced than, say, algebra, but this is true because we need algebra as a tool in order to work calculus.

You should rid yourself, at the beginning, of any superstition that some sort of unnatural reasoning power is required in order to understand calculus. This is simply not true. Every year thousands of average students satisfactorily complete courses in calculus.

1-4 Can learning be made easier and more certain? The process of learning calculus is not greatly different from that of acquiring other knowledge. For one thing, you must be serious about learning. Suppose that you have just acquired a new bit of information from an instructor or that you have read it in the textbook. You should ask yourself these questions:

1. Have I acquired this information *correctly?* Have I noted any limitations or exceptions attached to the statement I am learning?
2. Can I put this information into my own words, making the new wording agree in meaning with the original?
3. Can I think of any examples within my experience to which this information applies?
4. If the information is a new formula, rule, or theorem to be remembered, can I recite it aloud? Can I write it without looking at the text or notes?
5. Can I recite and write the information after an hour? A day? A week?

You should make a serious effort to learn and to understand each new piece of information as it is presented. On the other hand, you

cannot expect to be successful in this attempt in every case. Sometimes a seasoning process seems necessary, whereby the information takes on meaning with the passage of time; we become more convinced of the truth and the usefulness of information after we have pondered it and used it. William James, the psychologist, said that we seem to learn to swim in the winter and to skate in summer. Rather than become discouraged, then, if a new bit of information seems too much for you at first, you should at least "sleep on it" and find out whether or not you can master it the next day. A great French mathematician advised, "Go on, and faith will follow after you." George Chrystal, a famous mathematics teacher, suggested this variation: "Go on, but often return to strengthen your faith."

Unfortunately, the book has probably not yet been written which explains all essential parts of calculus in a manner completely understandable to every qualified student. It is generally necessary, therefore, for you to make use of some or all of the following suggestions:

1. Pay attention to the instructor's lecture material.
2. Read the text assignments thoroughly and, where necessary, reread.
3. It is often *necessary* to turn back to previously studied portions of mathematics books to get a fresh understanding of important topics. *Use the index.* Review portions of previous mathematics courses where necessary.
4. Obtain assistance and explanations, where needed, from instructors or from *successful students.* Out-of-class discussions serve a very useful purpose in all manner of courses. Getting explanations in words different from those used by the instructor or in the text is very useful.
5. Refer to other books. This method also makes use of the idea of getting explanations that are worded differently.
6. *Work the problems.* If you are satisfied with a superficial understanding and work no problems, you can be assured that your learning will be very temporary.

It almost goes without saying that a degree of mental discipline is required so that you will use an adequate amount of study time. It is unfortunate that some students who have adequate intelligence fail in calculus because of lack of application. Others, perhaps not so gifted intellectually, are more industrious, and they are likely to succeed.

1-5 Rules and definitions. The use of a certain number of rules and definitions is essential to calculus. You must understand and retain these essential statements.

Definitions are important because they ensure that the student, the instructor, and the author are talking about the same thing when a given mathematical term is used. An electronics specialist would not be able to work satisfactorily with another worker who, for example, used the word

"oscillator" to describe an amplifier, even though there may be some similarity between the two pieces of equipment. In the same way, it is essential to attach a specific meaning to each mathematical term.

Observe that we can define a term as indicating anything we please, so long as we actually do have a meaning attached to the term, and so long as we are consistent in the use of all our definitions. Naturally, we shall want our definitions to be those which are generally accepted by people who work with mathematics.

Rules, on the other hand, are established procedures which speed up our work. Just as we need procedures for neutralizing an amplifier or aligning a receiver, we require rules for quick and accurate use of our mathematical knowledge. Of course, an understanding of the logic behind the rules increases our ability to use the rules intelligently.

1-6 Can calculus be learned from this book alone? As in most courses, the inspiration and discipline of an instructor are powerful influences toward success. A sincere effort has been made, however, to include in this book the necessary explanations so that a qualified student can master the essentials of calculus by self-study.

It is sometimes worthwhile to refer to other calculus books to obtain the explanations of certain points in different words. Such reference books should be standard texts on calculus. They do not have to be the most difficult books, but poor results may be expected from books of the kind which oversimplify or popularize the subject.

There are many excellent calculus texts available. It would be improper here to recommend certain ones in preference to others. For the guidance of the student, however, we mention some examples of suitable texts:

1. H. M. Bacon: "Differential and Integral Calculus," 2d ed., McGraw-Hill Book Company, Inc., New York, 1955.
2. W. A. Granville, P. F. Smith, and W. R. Longley: "Elements of Calculus," Ginn & Company, Boston, 1946.
3. C. R. Wylie: "Calculus," McGraw-Hill Book Company, Inc., New York, 1953.

Instead of a strictly calculus reference, it may be advantageous to have a more complete treatment of college algebra, trigonometry, analytic geometry, and calculus. Such a unified treatment is provided in excellent form in these two books together:

4. F. L. Griffin: "Introduction to Mathematical Analysis," Houghton Mifflin Company, Boston, 1936.
5. F. L. Griffin: "Mathematical Analysis: Higher Course," Houghton Mifflin Company, Boston, 1927.

A reliable book of mathematical tables and data is particularly useful as a tool. The author has found the following books especially useful and recommends that you own one of them:

6. R. S. Burington: "Handbook of Mathematical Tables and Formulas," 3d ed., Handbook Publishers, Inc., Sandusky, Ohio, 1949.
7. C. D. Hodgman: "Mathematical Tables from Handbook of Chemistry and Physics," 9th ed., Chemical Rubber Publishing Co., Cleveland, Ohio, 1948.
8. E. S. Allen: "Six Place Tables," 7th ed., McGraw-Hill Book Company, Inc., New York, 1947 (a pocket-size book).

If you plan to make extensive use of your knowledge of calculus, you may wish to own a table of integrals. Standard books of this kind include:

9. H. B. Dwight: "Tables of Integrals and Other Mathematical Data," rev. ed., The Macmillan Company, New York, 1949.
10. B. O. Peirce: "A Short Table of Integrals," 3d ed., Ginn & Company, Boston, 1929.

A small book giving valuable suggestions for mathematics students is:

11. H. Dadourian: "How to Study, How to Solve: Arithmetic through Calculus," Addison-Wesley Publishing Company, Reading, Mass., 1949.

1-7 Arrangement of problems. Some textbooks are arranged so that a successful student should plan to work all, or nearly all, of the problems. This is *not* true of the present book.

Here, the problem lists usually contain two kinds of problems: *drill* exercises, using general symbols like x and y, and *applications*, where the quantities are voltage, current, time, etc. According to the arrangement used, the earlier drill problems are the simpler ones, while the later ones are often distinctly more difficult. Then come the application problems, the easier ones appearing first. The last few problems are often difficult enough to challenge the best students. The average student should not be expected to work all the problems provided in either category.

QUESTIONS

1. What are two basic types of problems which may be solved by calculus methods?
2. One who learns calculus acquires the ability to solve certain kinds of problems. What other advantage does he acquire?
3. In obtaining new mathematical information, what two principal kinds of material are used?
4. If you are not immediately able to comprehend a new piece of information in a calculus course, what procedure should you follow?
5. Why is it necessary to work a certain number of problems?
6. What is the reason for learning definitions in a mathematics course?
7. Why do we learn rules?

2

Functions

In this chapter we shall consider the nature of the quantities encountered in our work in electronics and the ways in which they may be related to each other.

2-1 Variables. A quantity which takes on different values during a given discussion or problem is called a *variable.*

In many applications, a variable may assume only certain kinds of values. For instance, the number of stages in an amplifier must be a real number; moreover, this number must be positive; and it must be an integer (1, 2, 3, etc.). On the other hand, the resistance of a circuit element does not have to be an integer, since it is entirely possible to have a resistor of 17.8 ohms, for example. With certain circuit arrangements, we can even obtain a negative resistance, as in a dynatron oscillator. Imaginary values of resistance are not considered (the imaginary component of impedance being called reactance).

The variables considered in this book are limited to real values unless a statement is made to the contrary.

In mathematical developments, variable quantities are usually represented by letters occurring toward the end of the alphabet, such as s, t, u, v, w, x, y, and z. The symbol t generally indicates the variable

time. Other symbols may represent other variables, in particular, the symbols for physical quantities such as length, current, voltage, force, etc.

2-2 Constants. A quantity which has an unchanging value throughout a particular problem or discussion is called a *constant.* Constants include natural numbers like 1, 2, 3, etc., fractions, irrational constants like $\sqrt{2}$, and the imaginary quantity $j = \sqrt{-1}$. The quantity π is an example of a constant, having a fixed value of approximately 3.1416. The base e of the natural system of logarithms has a constant value of approximately 2.718.

Symbols often used to represent constants include the earlier or middle letters of the alphabet, such as a, b, c, g, and h, or k, l, m, n, and p. The speed of light and of radio waves is represented by $c = 3 \times 10^8$ meters per second. The acceleration due to gravity is generally taken as a constant, symbolized by g. It has the value 32.2 feet per second per second, equivalent to 9.91 meters per second per second.

The symbols given in this and the preceding section are to be taken as constants or variables, respectively, only if nothing to the contrary is indicated. The symbol c, for instance, might be used in one problem to indicate the constant speed of light, while in another problem it might denote capacitance—perhaps a variable capacitance.

QUESTIONS

1. Define the term *variable.*
2. Define the term *constant.*
3. What are the values of two constants mentioned in Sec. 2-2?
4. Give five examples of symbols often used to represent constants in mathematical discussions. Do these symbols *always* represent constants?
5. Give five examples of symbols often used to represent variables in mathematical discussions. Do these symbols *always* represent variables?

PROBLEMS

In each of the following problems, identify the symbols representing constants and those representing variables. *Do not work the problems.*

1. The current i in a circuit is related to the applied voltage by the formula $i = v/R$, where $R = 100$ ohms. At what rate is i changing when the voltage v is increasing at a rate of 2 volts per second?

2. The distance s traveled by a freely falling object is given by $s = gt^2/2$, where g is the acceleration due to gravity and t is the time of fall in seconds. How fast is the object falling after 2 seconds?

3. In a certain circuit the current changed according to $i = I_0e^{-0.5t}$, where I_0 is the initial current in amperes, e is the base of the natural logarithmic system, and t is the time in seconds. How fast was the current changing after 50 microseconds?

4. A wooden beam can support a weight at its center given by $W = kbd^2/s$, where W is the weight in pounds, k is a constant depending upon the material of the beam,

b is the breadth of the beam in inches, d is the depth of the beam in inches, and s is the length of the beam in feet. What are the dimensions of the rectangular beam into which a log of 24 inches diameter should be cut in order to support the greatest weight? The beam is to be 14 feet long.

5. The distance s meters traveled in time t seconds by an electron starting from rest in an electric field is given by $s = e\mathbf{E}t^2/2m_e$, where e is the electron charge ($= 1.602 \times 10^{-19}$ coulomb), $\mathbf{E}$ is the electric-field intensity, and m_e is the mass of the electron ($= 9.106 \times 10^{-31}$ kilogram). How long will it take for an electron to move a distance of 0.1 meter, starting from rest, in a uniform field of 10^5 volts per meter?

2-3 Graphs. As a worker in electricity, you have doubtless made use of graphs. Thus it is a familiar fact that in a graph the points upon the horizontal axis indicate values of one quantity, while the height of the curve above the horizontal axis conveys the corresponding values of another quantity. An example is Fig. 2-1, which shows how the plate

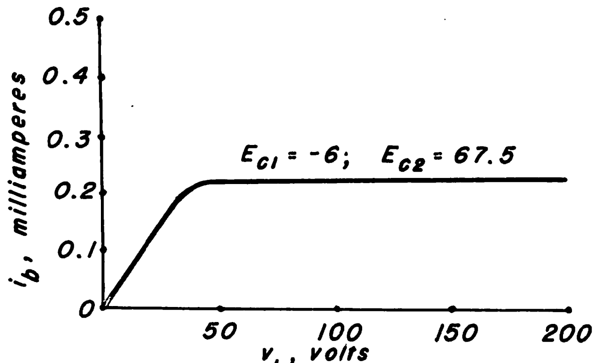


Fig. 2-1

current of a certain pentode changes with plate voltage. Here, the screen and control-grid voltages are assumed to be held constant. Many examples of useful graphs will occur to you. Some may be graphs which show variations of resonant frequency as the capacitance of a circuit is changed or variations in output voltage as the field current in a generator is varied.

In general discussions, horizontal distance is represented by x and vertical distance by y. A point P on a graph is identified by giving its x and y coordinates thus: $P(x,y)$. For example, a point 3 units to the right of the *origin* (crossing of the axes) and 2 units above might be indicated by $P(3,2)$.

2-4 Functions. We often read, in a problem or discussion, that "a variable y is a function of another variable x," or that "the plate current of a tube is a function of the plate voltage." Unless you have already learned a precise definition of the word "function," you probably have only a half-formed notion of the meaning of these statements.

Students sometimes get impressions like this: "If y is a function of x, then the value of y depends upon the value of x;" or perhaps they think "If y is a function of x, then the value of y changes when x is changed." For many cases, it is true, these ideas would be sufficient. To allow for certain exceptions, however, we adopt the following definition:

➢ If for each value of a variable x there is a corresponding value of y, then y is said to be a *function* of x.

Consider, as an example, the pentode whose graph of plate current versus plate voltage appears in Fig. 2-1. There is a value of plate current corresponding to each value of plate voltage shown. According to the above definition, then, this graph shows the plate current as a function of plate voltage.

Notice, however, that over a wide range of plate voltages (about 80 to 200 volts) the plate current *remains constant*, as far as our instruments can determine. In other words, the plate current in this portion of the graph does not change with, or depend upon, the plate voltage. The inexact ideas of the *function* relationship mentioned above become useless for a case like this, but the definition we adopted permits even a constant to be considered a function of some variable, and so it covers even this situation. The definition of *function* should be remembered.

QUESTIONS

1. If we are given that the current in a circuit is a function of the resistance of the circuit, what meaning is to be attached to that statement?
2. The charge in ampere-hours accumulated by a storage battery is said to be a function of the length of time during which it was charged. Interpret this statement.
3. The rf voltage appearing across a parallel-wire transmission line is a function of the distance of the point of measurement from the end of the line. What is the meaning of this statement?
4. A capacitor is connected through a resistor to a dc source. For each value of elapsed time there exists a value of charge in coulombs delivered to the capacitor. Express this statement in terms of the *function* definition.
5. State in *function* terms the fact that for each value of filament temperature in degrees a certain filament is capable of emitting some value of space current.
6. If a quantity y is a function of a variable x, and if x undergoes a change, does y necessarily change also?
7. Define the term *function*.
8. If for each setting of a certain capacitor in the range from 10 to 100 micromicrofarads the plate current of an associated electron tube has the value 50 milliamperes, could we still properly consider the plate current to be a *function* of the setting of the capacitor?

2-5 Dependent and independent variables. A variable to which we may arbitrarily assign any law of variation whatsoever is called an

independent variable. That is, an independent variable is one which may change in any manner whatsoever.

A *dependent variable*, on the other hand, is a function whose values correspond to those taken by some independent variable (or, in some cases, by more than one independent variable).

In graphing, the horizontal axis of the graph is commonly used to portray values of the *independent* variable. The height of the graph, corresponding to the distance along the vertical axis, is then used to indicate the corresponding values of the dependent variable as a function of the independent variable.

To illustrate independent and dependent variables, consider a circuit in which the applied emf is constant but the resistance is adjustable. We may vary the resistance in the circuit in any way we please, so that this quantity may be considered an independent variable. The current in the circuit, however, will assume values which in all cases correspond to the amount of resistance in the circuit, so that the current may be called a dependent variable. (It should be said here that we sometimes switch the dependent and independent variables around if it meets our mathematical needs to do so, regardless of which is actually the cause and which the effect.) The common practice, in considering general functions of a single variable, is to take x as representing the independent variable and y the dependent.

QUESTIONS

1. Define the term *independent variable.*
2. Define the term *dependent variable.*

PROBLEMS

In these problems a functional relationship exists between two variable quantities. In each case state which quantity you would consider the independent variable and which the dependent. State which quantity would be plotted along the horizontal axis and which along the vertical axis.

√1 **1.** The field current I of a generator is adjusted to control the output voltage V.

2. The power p radiated from an antenna varies with the current i supplied to the antenna.

3. A storage battery is charged at a constant current. The total charge, Q ampere-hours, depends upon the time t, in hours, during which the charge was carried on.

4. A signal generator is connected to the input circuit of a discriminator. As the frequency f of the input signal is varied, the output voltage v of the discriminator changes.

5. The reactance X_L of a coil changes as the applied frequency f is varied.

6. The resonant frequency f of a tuned circuit is adjusted by changing the amount of capacitance C in the circuit.

7. The permeability μ of a sample of iron depends upon the flux density **B**.

(1) I is the Indep. Var. plotted on x axis

V is the Dep Var. " " y axis

2-6 Functional notation. To indicate that a quantity y is a function of a variable x we write

$$y = f(x) \tag{1}$$

This is read "y equals the f function of x," or simply "y equals f of x." The meaning, of course, is that y is some function of x; that is, y has a value for each value of x.

This *functional notation* is widely used because it is brief and convenient. (Note that this symbolism does *not* mean that some quantity f is to be multiplied by some quantity x.)

The functional notation does not indicate, of itself, just *what* function of x is represented by y. For example, the statement $y = f(x)$, if nothing further is given, can mean that $y = x$, or that $y = 2x^2$, or perhaps that $y = \sin x$.

In a given problem or discussion we may wish to consider *more than one function* of some variable. To do so we provide similar notations to suit our purpose. For instance, $g(x)$, which is read "the g function of x," might be used, or $F(x)$, "the capital-F function of x," could be considered. As concrete examples, suppose that we are dealing with a situation in which y is taken to be the f function of x and that y is known to be equal to the tangent of x. Also suppose that a variable u, taken as the g function of x, has been determined to be equal to x^3 and that an additional variable v, termed the F function of x, equals $5x$. To indicate the foregoing we can write

$$\begin{aligned} y &= f(x) = \tan x \\ u &= g(x) = x^3 \\ v &= F(x) = 5x \end{aligned}$$

An important rule which is applied to the use of the functional notation is this:

Throughout a single problem or discussion, a given functional symbol such as $f(\ )$, represents the same operation or operations performed on whatever expression appears in the parentheses of the functional symbol.

Example. If we are given that $f(x) = x^2 - 9x$, then in the same discussion, we should take $f(y)$ to indicate $y^2 - 9y$; likewise, $f(z)$ would indicate $z^2 - 9z$.

An application of the above rule follows. Suppose we know that the value of some function, such as $f(x)$, is 5 when x is equal to 3. We may write this statement briefly:

$$f(3) = 5$$

This practice is described as follows. To *indicate* the value of a function of x, which we shall call $f(x)$, when x takes a particular value a, write

$f(a)$. The value of $f(a)$ is obtained by substituting a for every x in the expression for $f(x)$. As an example, suppose that $f(x) = 60x^4 + x + 1$. To find $f(2)$, we would substitute 2 for each x in this expression, getting $f(2) = 60(2)^4 + 2 + 1 = 963$.

QUESTIONS

1. Why is functional notation used?
2. What information is contained in the statement $y = f(x)$?

PROBLEMS

In Probs. 1 to 3 read the given expressions aloud.

1. $y = f(w)$ **2.** $u = s(r)$ **3.** $x = h(t)$

In Probs. 4 to 9 write the functional expressions for the given statements.

4. y equals f of x
5. v equals g of u
6. y equals capital G of x
7. z is the f function of w → $z = f(w)$
8. u equals the g function of v
9. r equals the s function of t → $r = s(t)$

10. If $f(x) = x^3 + 2$, express $f(y)$.
11. If $g(x) = \sin 3x$, express $g(w)$.
12. If $h(x) = 130 \cos x$, express $h(r)$.
13. If $f(x) = 3x^3$, find $f(3)$.
14. If $f(y) = y^5 + 1{,}000$, find $f(10)$.
15. If $g(z) = 0.01z^2$, find $g(0.1)$.
16. If $G(x) = 2 - \cos x$, find $G(0)$.

13
$f x = 3x^3$
$f(3) = 3(3)^3$
$= 3(27)$
$f(3) = 81$

2-7 Single-valuedness and continuity of functions. Consider the curve of Fig. 2-2. The equation for this curve is $y^2 = x$ (and the curve is an example of a *horizontal parabola*). Solving the given equation for y, we have $y = \pm\sqrt{x}$. For each positive value of x, then, we have *two* values of y, one positive and one negative. We note that our definition of the function relationship does not prevent y from having *more than one value* for some or all values of x. Here, y is a two-valued function.

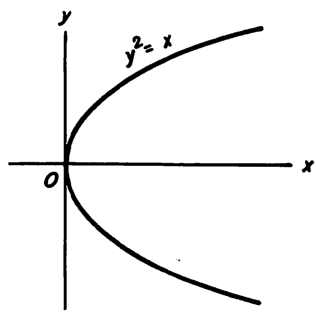


Fig. 2-2

Functions can be classed as either *single-valued*, that is, functions having only one value for each value of their independent variable, or *multivalued*, that is, functions having two or more values for each value of the independent variable. The plate-current curve of Fig. 2-1 illustrates a single-valued function, while the function of Fig. 2-2 is not single-valued.

In some kinds of problems functions like that of Fig. 2-2 are handled by treating the two branches of the graph separately; that is, we might first consider the function $y = \sqrt{x}$, representing the upper branch* of the curve, and afterward we could consider the lower branch, represented by $y = -\sqrt{x}$. But our treatment in this book will generally be restricted to single-valued functions.

Functions may be further classed as either *continuous* or *discontinuous*. For our purposes it is perhaps sufficient to think of a continuous function as one whose graph is free of (*a*) breaks or open regions or (*b*) abrupt vertical "jumps" in value. Breaks or jumps in a graph are called *discontinuities*, and a function whose graph includes them is called *discontinuous*.

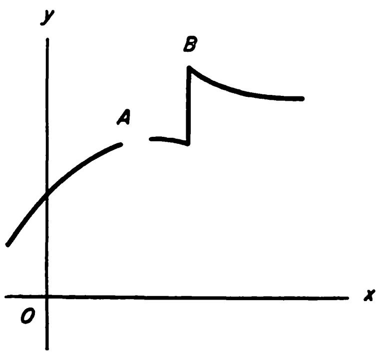


Fig. 2-3

The plate-current graph of Fig. 2-1 illustrates an example of a continuous function. But the graph of Fig. 2-3 has a *break* at A and an abrupt *jump* at B, so we say that the function graphed in Fig. 2-3 is discontinuous at these portions of the graph. Functions representing electric or other natural phenomena are, for the most part, continuous. For the purposes of this book the term *function* will mean a *single-valued, continuous* function unless otherwise stated.

QUESTIONS

1. What is a single-valued function?
2. What is a multivalued function?
3. If $x^2 - y^2 = 16$, and if $x = 5$, how many corresponding values are there for y?

In questions 4 to 8 state which of the functions are single-valued and which are multivalued.

4. The cosine of an angle expressed as a function of the angle.
5. The flux density in a saturable iron core expressed as a function of the intensity of an alternating magnetizing field (see a *hysteresis curve*).
6. The power dissipated in a resistor expressed as a function of the current flowing in it.
7. The plate current of a tube (working into a reactive load and having an elliptical load line) expressed as a function of the plate voltage.
8. The function $\sin^{-1} x$ expressed as a function of x.

9. What is a continuous function?

* Note that $\sqrt{x}$, unless preceded by a minus sign, indicates the *positive* square root of x.

10. What is a discontinuous function?
11. Give an example of a continuous function.
12. Is the function $y = x$ continuous?
13. Give an example of a discontinuous function.
14. Are functions representing physical quantities, in general, continuous?

2-8 Explicit and implicit functions. Thus far we have considered some cases in which the dependent variable y was expressed as an *explicit function* of the independent variable x. That is, a functional relationship $y = f(x)$ has been given in each case, so that it is only necessary to substitute particular values of x in the resulting formula in order to obtain the corresponding values of y. For example, if we are given that $y = 2x^2 + 9$, and if we wish to know the value of y when $x = 1$, it is only necessary to substitute 1 for x in the expression for y, getting $y = 2(1)^2 + 9 = 11$.

But the functional relationship between y and x might be of a different kind. In a series resistive-reactive circuit having a given impedance Z, the resistance R and the reactance X are related by

$$X^2 + R^2 = Z^2$$

or

$$X^2 + R^2 - Z^2 = 0$$

Here the functional relationship is not of the form $R = f(X)$ but rather of the form $F(R,X) = 0$. This may be read "the capital-F function of R and X is equal to zero."

In the case of a function y of a variable x, a similar expression might be, for instance, $x^2 + y^2 - 25 = 0$. And the statement that such a functional relationship exists would be

$$F(x,y) = 0 \tag{2}$$

In cases like the above, y is said to be an *implicit function* of x.

Sometimes it is a simple matter to convert a given implicit functional relationship into an explicit one. In the case above, for instance, we can solve for y, getting $y = \pm \sqrt{25 - x^2}$. On the other hand, it will sometimes be difficult or even impossible to solve a given implicit relationship to obtain an explicit one. There are, however, ways of applying calculus methods to such functions. In higher courses it is proved that under appropriate conditions a relationship of the form $F(x,y) = 0$ establishes y as a function of x.

QUESTIONS

1. What equation, in functional notation, indicates that y is given as an explicit function of x?

2. Give an equation, in functional notation, which indicates that y is an implicit function of x.

3. Give an example of an explicit functional relationship between y, a dependent variable, and x, an independent variable.

4. Express the power p in a circuit as an explicit function of the voltage v and the current i.

5. Express the power p in a circuit as an explicit function of the voltage v and the resistance r.

6. Give an equation, in functional notation, indicating that some function of x and y equals zero.

7. Give an example of an implicit functional relationship between two quantities x and y.

8. Express the following relationships in such a way that y is given as an explicit function of x in each case: (a) $x^2 + y^2 = 100$; (b) $x - y^2 = -a$; (c) $y/x - x - y = 0$.

9. Is it always possible to rewrite an expression of the form $F(x,y) = 0$ to obtain a relationship of the form $y = f(x)$?

2-9 Function of a function. Suppose that a variable such as p (the instantaneous power in a resistor of resistance R) is known to have a functional relationship to some other variable such as i (the instantaneous current in the resistor). In this case, the relationship is $p = i^2R$.

Figure 2-4 is a graph of the above equation. Now, suppose that i, in turn, is a function of some other variable, such as time t. For instance,

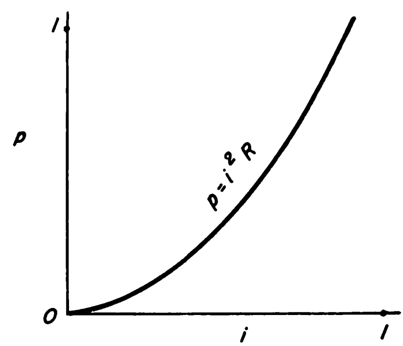


Fig. 2-4

we may be given that $i = 1/t$, as graphed in Fig. 2-5. If we desire to express p directly in terms of t, then we may do so simply by substituting $i = 1/t$ in the first equation, getting $p = (1/t^2)R = R/t^2$. Note that for this substitution it is necessary to have formulas both for p as a function of i and for i as a function of t. The resulting curve for the function $p = R/t^2$ is given in Fig. 2-6.

Here it becomes clear that for each value of t representing a time during which the current flows there will also exist a value of p. Thus, in addition to being a function of i, the dependent variable p is also a function of t. We say that p is a function of t through the variable i.

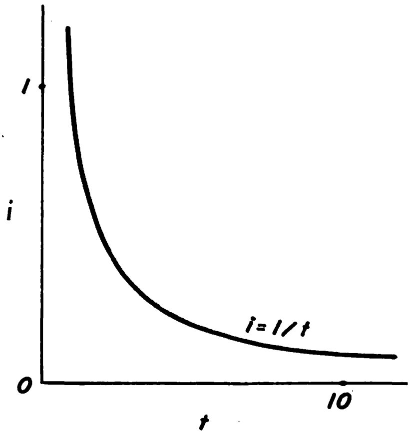


Fig. 2-5

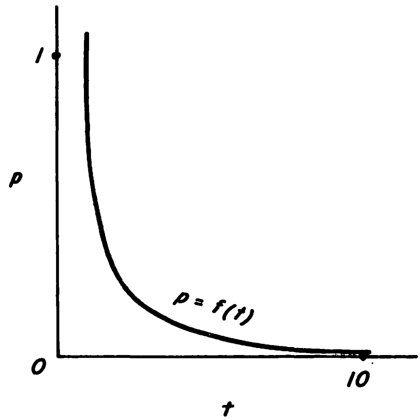


Fig. 2-6

In a case of this kind p is called a *function of a function.* Functions of functions are sometimes called *composite functions.*

QUESTIONS

1. If $y = 2x$, and if $x = s^3$, express y as a function of s.
2. If $p = v^2/r$, where $r = 100$ ohms, and if $v = 10t - t^2$, find the value of p when $t = 5$ seconds.
3. If we know A to be a function of B, and if we know B to be a function of C, what two pieces of information do we need in order to express A directly as a function of C?

① $y = 2x$

$x = s^3$

$y(s) = 2s^3$

② $p = \frac{v^2}{r}$ $t = 5$ sec $R = 100\,\Omega$

$v = 10t - t^2$

$v = 10(5) - (5)^2 = 25$

$p = \frac{(25)^2}{100} = 6.25$ w

③ $A = f(B)$

$B = g(c)$

$A = f(g(c))$

3

Average Rates

One of the most important classes of problems dealt with in calculus is that of the exact rate of change of a quantity. In the present chapter we prepare the way for the handling of this problem by treating *average* rates and related subjects.

3-1 Increments. Expressed in the simplest terms, an *increment* is a change in the value of a quantity.

The symbol Δ (Greek capital letter delta) is used to indicate an increment. If we write Δx, read "delta x," it is taken to mean an *increment* or *change* in x. In other words, Δx is a symbol representing the amount of change taking place in the value of a quantity x, just as x represents the value of the quantity itself.

Note that the symbol Δx does *not* indicate that some quantity Δ is multiplied by some other quantity x. Specifically, Δx is a single quantity, the change in x.

In the same way, Δy indicates an increment or change in the value of y, while Δt and Δz represent, respectively, amounts of change in t and in z.

To help fix the idea of an increment, let us refer to Fig. 3-1. This graph shows the functional relationship between two variables, x and y.

Suppose that x has some particular value, say x_1. Now, if x changes from x_1 to some new value, say x_2, we refer to the amount of this change as Δx. Briefly, $\Delta x = x_2 - x_1$.

While x changes from x_1 to x_2, y changes from y_1 to y_2, as shown. The increment in y, Δy, is then equal to $y_2 - y_1$.

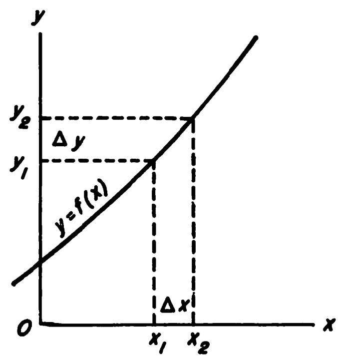


Fig. 3-1

An *increase* in the value of a quantity is indicated by a *positive increment*, while a *negative increment* indicates that the quantity has undergone a *decrease* in value. A constant, of course, has no increment other than zero.

Increments are, in general, variables. We may consider increments in *various* quantities, and these changes may be either large or small.

QUESTIONS

1. What is the meaning of the term *an increment of x?*
2. What is the meaning of the symbol Δx? Of Δy?
3. If the sign of an increment in y is positive, what does this fact indicate? What is indicated by a negative increment? An increment equal to zero?

PROBLEMS

1. If $y = x^2 + 3x - 2$, and if x has an original value equal to 1, find the value of Δy corresponding to a Δx of 1.

2. If $y = x^3 + 2x$, find the change Δy which occurs in y as x is varied from $x = 2$ to $x = 3$.

3. If $y = 2x^5 + x^4 - 3x^3 - 5x^2 - x + 1$, find the value of the change Δy which takes place in y as x changes from 0 to 2.

4. Given that $\omega = 2\pi f$, what change in ω results from a change in f from 100 to 120 cycles per second?

5. The voltage applied to a circuit is changed from 250 to 275 volts. What is the value of Δv? Given that $i = v/R$ and that $R = 10$ ohms, what Δi corresponds to the above voltage change?

6. If $p = i^2R$, and if $R = 100$ ohms, find Δp if i changes from 1 to 2 amperes.

7. The current i in a circuit varied with time t according to $i = 10t^2 + 5t$. During the interval from $t = 1$ to $t = 2$ seconds what was Δi?

8. The electric charge transferred from a capacitor changed according to $q = 10^{-5}t^2$. As t changed from 5 to 6 seconds what was the value of Δq?

9. A 1,000-kilocycle voltage is applied to a capacitor of 500 micromicrofarads. If the frequency is changed by an amount $\Delta f = -500$ kilocycles, find ΔX_C.

10. The gain of a certain amplifier stage is represented approximately by $A_v = g_mR_L$. If $g_m = 6 \times 10^{-3}$ mho, and if $R_L = 2{,}200$ ohms, how much change ΔR_L should be made in R_L to produce a change $\Delta A_v = 3$?

11. The resonant frequency of a circuit varies according to $f_r = 1/2\pi\sqrt{LC}$. Given that $L = 100$ microhenrys and $C = 400$ micromicrofarads, what change Δf_r would occur in the resonant frequency if L were changed by an amount $\Delta L = 21$ microhenrys?

12. Figure 3-2 shows how the magnetic flux density **B** in a sample of iron changes as a function of field intensity **H**. If the permeability of the sample is $\mu = \mathbf{B}/\mathbf{H}$, describe how to find graphically the point of greatest permeability.

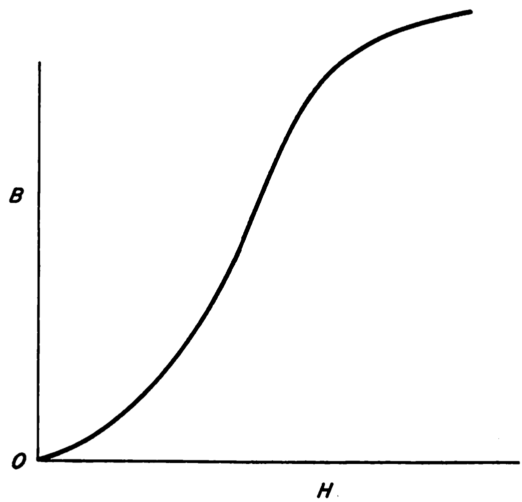


Fig. 3-2

3-2 Some applications of increments.

Certain tube characteristics can be defined (at least in an elementary way) as the ratios of *small* changes, or increments, in the voltages and currents of the tube elements. We can get the plate resistance of a tube approximately by the formula

$$r_p = \frac{\Delta v_b}{\Delta i_b} \quad \text{ohms} \tag{1}$$

where v_b is the plate voltage and i_b is the plate current. The grid voltage v_c is assumed to be held constant here.

Consider a tube whose characteristics are shown in Fig. 3-3. For a grid voltage $v_c = -10$ volts we find that a plate voltage of 300 volts causes a plate current of 10.2 milliamperes to flow. If the plate voltage is raised to 320 volts, a new value of plate current, 13.0 milliamperes, is obtained. In increment notation, the plate voltage change is $\Delta v_b = 20$ volts, while the plate current change is

$$\Delta i_b = 2.8 \text{ milliamperes} = 0.0028 \text{ ampere}$$

Substituting these values in Eq. (1) above, we get $r_p = 7{,}140$ ohms.

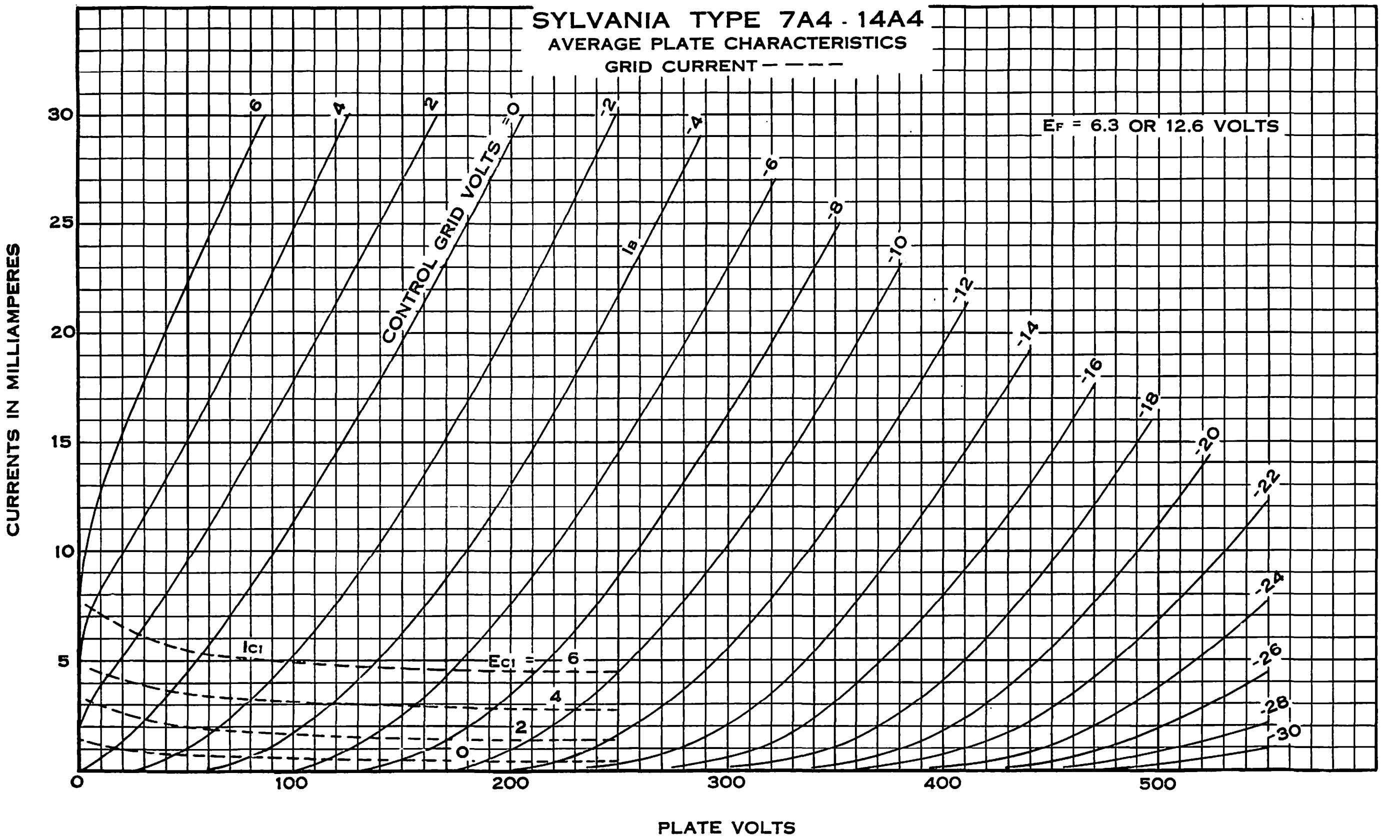

SYLVANIA TYPE 7A4 - 14A4
AVERAGE PLATE CHARACTERISTICS
GRID CURRENT - - - -
EF = 6.3 OR 12.6 VOLTS
CONTROL GRID VOLTS = 0
6
4
2
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
-22
-24
-26
-28
-30
IB
IC1
EC1 = 6
4
2
0
CURRENTS IN MILLIAMPERES
30
25
20
15
10
5
0
100
200
300
400
500
PLATE VOLTS

Fig. 3-3

The mutual conductance of a tube is given approximately by

$$g_m = \frac{\Delta i_b}{\Delta v_c} \quad \text{mhos} \tag{2}$$

where v_b is assumed constant. In Fig. 3-3 we have seen that for $v_b = 300$ volts and for $v_c = -10$ volts we get a current $i_b = 10.2$ milliamperes. It is also easily found from the graph that, when $v_b = 300$ volts and $v_c = -8$ volts, the plate current is 16.2 milliamperes. Thus

$$\Delta i_b = 6 \text{ milliamperes} = 0.006 \text{ ampere}$$

while $\Delta v_c = 2$ volts. Substituting these values in (2) gives

$$g_m = 3 \times 10^{-3} \text{ mho} = 3{,}000 \text{ micromhos}$$

Similarly, we can get the amplification factor of a tube approximately from

$$\mu = -\frac{\Delta v_b}{\Delta v_c} \tag{3}$$

keeping i_b constant. From the graph, we find that, if we increase the plate voltage of the tube from 300 to 340 volts and change the grid voltage from -10 to -12 volts, the plate current remains unchanged at 10.2 milliamperes. For a constant i_b, then, we may take $\Delta v_b = 40$ volts and $\Delta v_c = -2$ volts. These values substituted in (3) give $\mu = 20$.

Similar considerations can be used in stating the characteristics of transistors. For instance, letting i_c indicate the collector current and i_e the emitter current, we can express the *current gain* of a transistor as

$$\alpha = \frac{\Delta i_c}{\Delta i_e} \tag{4}$$

where the collector voltage v_c is retained constant. Other characteristics, such as mutual conductances, are often expressed for transistors in much the same way as for tubes.

3-3 Average rates. Suppose we know that a quantity y is a function of a variable x. As an example, consider the functional relationship illustrated in Fig. 3-4. Note that, as x changes from x_1 to x_2, y changes from y_1 to y_2. The symbols Δx and Δy are again used to indicate the increments in x and y, respectively.

The average rate of change of y with respect to x is defined as the change in y divided by the change in x,

or

$$\text{Average rate of change of } y \text{ with respect to } x = \frac{\Delta y}{\Delta x} \tag{5}$$

If y and x indicate physical quantities, $\Delta y/\Delta x$ indicates the average

number of units of increase in y *for each unit* of increase in x over the interval Δx. If, for instance, we consider a voltage v which changes as a function of time t, then $\Delta v/\Delta t$ might show the average number of volts gained for each second of time as time progressed over the interval Δt. Such an average rate could be expressed as so many *volts per second.*

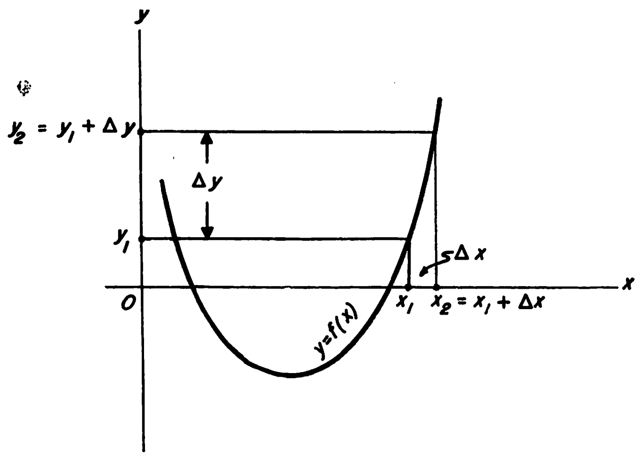


Fig. 3-4

Example. The output voltage v of a certain generator "builds up" with time, so that 2 seconds after starting, the voltage is 100 volts, becoming 155 volts after 13 seconds of operation. What is the average rate of change of voltage with respect to time over the interval from $t = 2$ to $t = 13$ seconds?

Substituting in (5),

$$\text{Average rate of change of } v \text{ with respect to } t = \frac{55}{11} = 5 \text{ volts per second}$$

In many cases, instead of numerical values of x and y, we may be given an *equation* which relates these quantities. For example, we might be given that

$$y = x^2 - 4x + 4 \tag{6}$$

In such cases, we can obtain a *new* equation expressing the *average rate of change* (of y with respect to x) as a function of x and Δx. An example follows.

Let it be desired to find (*a*) a formula for the average rate of change of y with respect to x if y follows equation (6), and (*b*) the value of this average rate of change from $x = 1$ to $x = 4$.

If, in (6), we let $x = x_1$, we get an equation for y_1:

$$y_1 = x_1^2 - 4x_1 + 4 \tag{7}$$

Likewise, letting x change to a new value $x_1 + \Delta x$, we obtain an equation

for a new value of y:

$$y_1 + \Delta y = (x_1 + \Delta x)^2 - 4(x_1 + \Delta x) + 4$$

or
$$y_1 + \Delta y = x_1^2 + 2x_1\,\Delta x + \Delta x^2 - 4x_1 - 4\Delta x + 4 \tag{8}$$

Subtracting (7) from (8), we get an expression for Δy:

$$\begin{array}{rl} y_1 + \Delta y &= x_1^2 + 2x_1\,\Delta x + \Delta x^2 - 4x_1 - 4\Delta x + 4 \\ y_1 \quad &= x_1^2 \qquad\qquad\qquad\qquad - 4x_1 \qquad\quad + 4 \\ \hline \Delta y &= \qquad 2x_1\,\Delta x + \Delta x^2 \qquad - 4\Delta x \end{array}$$

If this result is divided by Δx, we get the desired formula for $\Delta y/\Delta x$:

$$\frac{\Delta y}{\Delta x} = 2x_1 + \Delta x - 4 \tag{9}$$

This satisfies part (a) of our project. To complete part (b) let $x_1 = 1$ and $\Delta x = 3$ in (9), getting

$$\frac{\Delta y}{\Delta x} = 2(1) + 3 - 4 = 1$$

That is, over the interval from $x = 1$ to $x = 4$, y increased *on the average* 1 unit for each unit of increase of x.

You should carefully remember the simple sequence of steps used to obtain the formula for $\Delta y/\Delta x$:

1. In the given equation, let $x = x_1$, getting an equation for y_1.
2. Again in the given equation, let $x = x_1 + \Delta x$, getting an equation for $y_1 + \Delta y$.
3. Subtract the equation for y_1 from the equation for $y_1 + \Delta y$, obtaining an equation for Δy.
4. Divide each term of the equation for Δy, by Δx. This results in the desired equation for $\Delta y/\Delta x$.

QUESTIONS

1. Define the term *average rate of change* of a function y with respect to a variable x.
2. Express the definition required in question 1 as an equation.
3. If the value of y increases as x increases, will the average rate of change of y with respect to x be positive or negative?
4. If the value of y decreases as x increases, will the average rate of change of y with respect to x have a negative or a positive value?
5. State the four steps necessary to obtain an equation for $\Delta y/\Delta x$ if we are given an equation for y as a function of x.

PROBLEMS

1. If $y = x^2 + 2x$, find a formula for $\Delta y/\Delta x$. What is the value of $\Delta y/\Delta x$ if x changes from 2 to 2.3?

2. Given that $y = 3x^2 + 9$, obtain a formula for $\Delta y/\Delta x$. Find the average rate of change of y with respect to x as x changes from 0 to 1.

3. If $y = 2x^2 + x + 100$, find a formula for $\Delta y/\Delta x$. Find the average rate of change of y with respect to x as x changes from 10 to 20.

4. If $y = x^3 - x^2 + 2x + 20$, what formula gives $\Delta y/\Delta x$? Find $\Delta y/\Delta x$ as x varies from 1 to 2.

5. The reactance of an inductor varies according to $X_L = 2\pi fL$. If $f = 1{,}000$ cycles per second, find the average rate of change of X_L with respect to L as L changes from 0.2 to 2 henrys.

6. The current in a circuit is given by $I = V/R$. Find the average rate of change of I with respect to V as V changes from 5 to 7 volts if $R = 20$ ohms.

7. As in Prob. 6, but find the average rate of change of I with respect to R as R changes from 10 to 12 ohms if $V = 20$ volts.

8. The current in a circuit followed the formula $i = 10 - 10t^2$ amperes, where t was in seconds. Give a formula for the average rate of change of current in amperes per second. What was the average rate of change from $t = 0.001$ to $t = 0.002$ second?

9. Given that $p = i^2R$, find the average rate of change of power with respect to current as i changes from 2 to 4 amperes if $R = 6$ ohms.

10. If $p = v^2/r$, find the average rate of change of power with respect to resistance as r varies from 5 to 10 ohms. Let $v = 10$ volts.

11. The reactance of a capacitor varies according to $X_C = 1/2\pi fC$. Find the average rate of change of X_C with respect to f for a 0.0001-microfarad capacitor as f is varied from 900 to 1,100 kilocycles.

12. Figure 3-5 is a graph of the voltage v across a certain transmission line as a function of distance s, in miles, along the line from the sending end. Suppose that

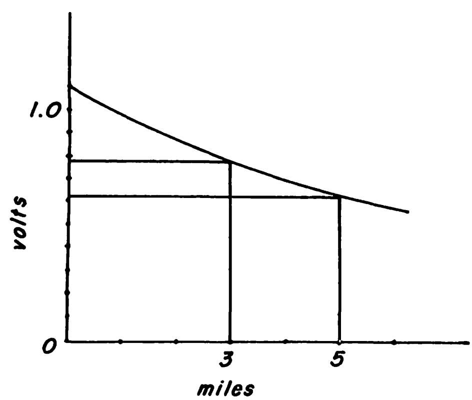
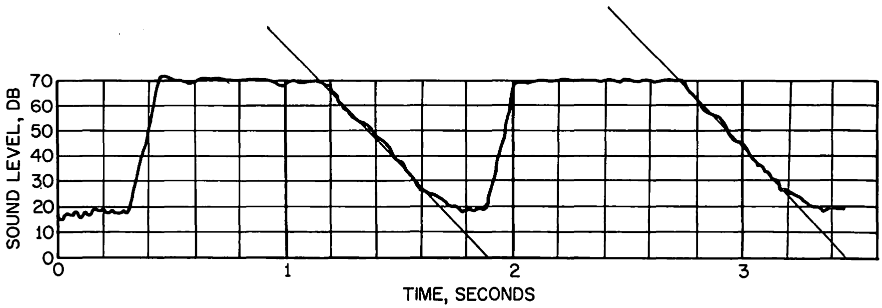


Fig. 3-5

$v = 0.62$ volt at a point 5 miles from the source and that $v = 0.78$ volt at a distance 3 miles from the source. Find $\Delta v/\Delta s$ over this range. Explain the significance of the minus sign in the result.

13. The attenuation of a certain low-pass filter increased from 7 decibels at 400 cycles per second to 29 decibels at 1,600 cycles per second. (NOTE: The original frequency, 400 cycles per second, has been multiplied by 4 to give the new frequency, 1,600 cycles per second. Since doubling the frequency is equivalent to increasing it by *one octave*, the frequency here has undergone a change of two octaves.) Find the average rate of change of attenuation in decibels per octave.

14. Figure 3-6 illustrates how a 1,024-cycle sound died away during two separate tests of the reverberation characteristics of a broadcast studio. Taking indications from the straight lines drawn over the graph, find the average rate in decibels per second at which the sound died away.



Courtesy of Stanton D. Bennet

Fig. 3-6

15. Given that $y = f(x)$, show that (using the functional notation) the average rate of change of y with respect to x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

3-4 Some electrical quantities.

a. Average current. You know about certain electrical quantities which are defined in terms of average rates of change. The average value of a *current*, in amperes, flowing past a point in a circuit is defined as the average rate of movement of charge past the point, in coulombs per second. Symbolically,

$$I_{av} = \frac{\Delta q}{\Delta t} \quad \text{amperes} \tag{10}$$

Example 1. After a circuit was closed for 13 seconds, a total charge of 33 coulombs had been conducted past a certain point. After the circuit was closed for 35 seconds, the total charge transmitted had increased to 165 coulombs. Find the average current in the circuit from $t = 13$ to $t = 35$ seconds.

Substituting in (10) gives

$$I_{av} = {}^{132}\!/_{22} = 6 \text{ coulombs per second} = 6 \text{ amperes}$$

b. Inductance. The property of a circuit called *inductance* (to be defined more adequately later in this book) can be described as the ability of a circuit to develop an induced emf as a result of a *change* of current in the circuit. If, in a certain circuit, the current changes at an average rate $\Delta i/\Delta t$ amperes per second, and if the resulting average induced voltage is V_{av} volts, then the inductance of the circuit is given by

$$L = -\frac{V_{av}}{\Delta i/\Delta t} \quad \text{henrys} \tag{11}$$

Example 2. The current in a coil was 1.1 amperes at a certain time, which we shall call $t = 0$. Three seconds later the current had become 9.5 amperes. During this period of time an average voltage of -42 volts was induced as a result of the current change. Find the inductance of the circuit.

Substituting in (11),

$$L = -\frac{V_{av}}{\Delta i/\Delta t} = -\frac{-42}{8.4/3} = 15 \text{ henrys}$$

c. Current in a capacitor. As another application of average rates, it can be shown that, when the voltage v across a capacitor is varied, the average current into the capacitor over a time interval Δt is

$$i_{av} = C\frac{\Delta v}{\Delta t} \qquad \text{amperes} \tag{12}$$

Example 3. A voltage $v = 200 + t^2$ was applied to the plates of a capacitor. What was the average current into the capacitor over the interval from $t = 3$ to $t = 5$ if $C = 10$ microfarads?

A formula for $\Delta v/\Delta t$ is obtained by the calculation

$$\begin{aligned} v_1 &= 200 + t_1^2 \\ v_1 + \Delta v &= 200 + (t_1 + \Delta t)^2 \\ &= 200 + t_1^2 + 2t_1\,\Delta t + \Delta t^2 \end{aligned}$$

Subtracting,

$$\Delta v = 2t_1\,\Delta t + \Delta t^2$$

so

$$\frac{\Delta v}{\Delta t} = 2t_1 + \Delta t$$

If $t_1 = 3$ and $\Delta t = 2$ and if $C = 10^{-5}$ farad, Formula (12) gives

$$i_{av} = 10^{-5}(6 + 2) = 8 \times 10^{-5} \text{ ampere}$$

QUESTIONS

1. Define the term *average current* in terms of an average rate of change.
2. Define the term *inductance* in terms of the average induced voltage in a circuit and the average rate of change of current.
3. Give a formula for the average current flowing into a capacitor in terms of the rate of change of applied voltage.

PROBLEMS

1. If the charge q coulombs stored in a capacitor varied according to $q = 0.0001 + 0.01t^2$, where t was in seconds, what average current I_{av} was useful in charging the capacitor from $t = 0$ to $t = 0.1$ second?

2. If a charge of 120 ampere-hours was given a storage battery during an 8-hour charging period, what was the average current in amperes used in charging the battery?

3. A battery charger can supply an average charging current of 6 amperes to a certain battery. How long will it take to deliver a charge of 800 ampere-hours?

4. The current through a coil having an inductance $L = 10$ henrys varied as $i = 10 - 3t$ amperes. What average voltage V_{av} was induced in the coil from $t = 0$ to $t = 1$ second?

5. Find the average voltage appearing across an inductor of 30 henrys during the time from $t = 0$ to $t = 2$ seconds when a current $i = 2(t^3 - t^2)$ is sent through the inductor.

6. The current in a certain coil was $i = 5 - t^2$ amperes. What was the inductance of the coil if the average induced emf, from $t = 1$ to $t = 4$ seconds, was -5 volts?

7. The voltage applied across a capacitor changed at an average rate of 1,000 volts per second. If the resulting average current was 5 milliamperes, what must be the capacitance?

8. A 2-microfarad capacitor was subjected to a voltage $v = 1{,}000 - t^3$ volts. Find the average current from $t = 0$ to $t = 10$ seconds.

9. The voltage applied to a capacitor varied as $v = 2t^3 - 4t^2 + t + 2$ volts. Find the capacitance if the average current from $t = 0$ to $t = 2$ seconds was 4 microamperes.

3-5 Average rates treated graphically. Figure 3-7 shows how a certain quantity y varies as a function of x. Let x change from an original value x_1 to a new value x_2, and call the amount of this change Δx,

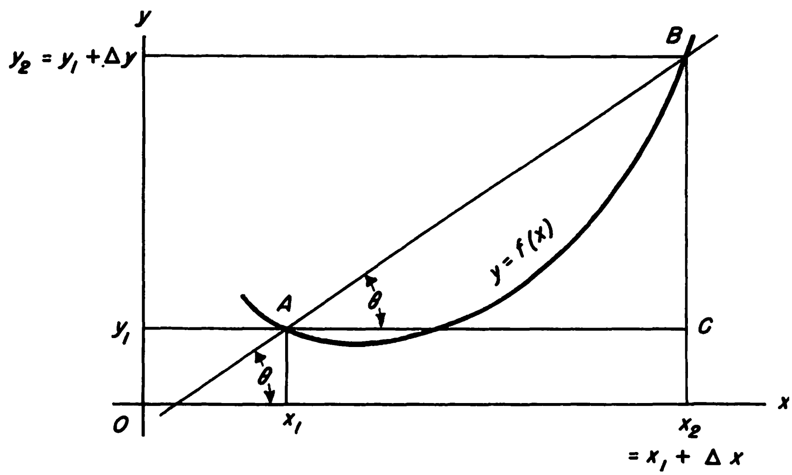


Fig. 3-7

as indicated. Then y will undergo a change from y_1 to y_2, and we call this change Δy. As already defined, the average rate of change of y with respect to x over this range is $\Delta y/\Delta x$.

Consider a point A on the curve, where $x = x_1$ and $y = y_1$. Let us draw a straight line AB from this point through point B, where $x = x_2$ and $y = y_2$. Then line AB is a secant line to the curve through points A and B. (A *secant line* to a curve is a straight line cutting the curve in two or more points.)

Now consider the angle θ between the secant line AB and a horizontal line AC through A. The tangent of this angle is given by

$$\tan\,\theta = \frac{CB}{AC}$$

But, as shown in the figure, $CB = \Delta y$, and $AC = \Delta x$. Therefore

$$\tan\,\theta = \frac{\Delta y}{\Delta x} \qquad (13)$$

Thus the average rate of change of y with respect to x, while *defined* as $\Delta y/\Delta x$, is also equal to the tangent of the angle θ.

Note that, since AC is parallel to the axis OX, the angle θ is formed also between AB (extended) and OX. The quantity $\tan\,\theta = \Delta y/\Delta x$ is the *average slope* of the graph over the interval from x_1 to x_2.

In the special case in which the graph is a straight line, as in Fig. 3-8, the rate of change of y with respect to x is constant. In such a case, the average rate of change of y with respect to x will be given, for any interval, by the tangent of the angle θ formed between the graph itself and the axis OX. Thus a straight-line graph has a *constant slope.*

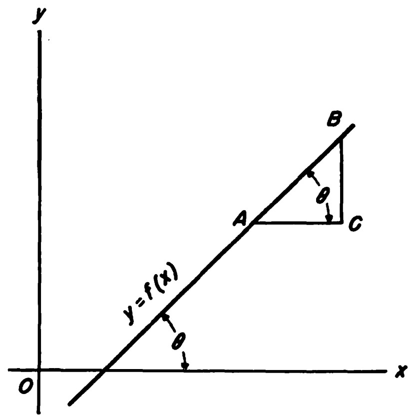


Fig. 3-8

QUESTIONS

1. What is a secant line to a curve?
2. What is a definition of the term *average slope of a curve?*
3. What is the relationship between the average slope of the graph of a function and the average rate of change of that function?
4. Give the relationship between the average slope of a curve between two points and the slope of the secant line to the curve through these points.
5. Is the *slope of a straight-line graph* a constant or a variable?

PROBLEMS

1. If $y = 2x^3 + x$, what are the values of Δx and of Δy as x changes from $x = 1$ to $x = 3$? What is the value of the average rate of change of y with respect to x over this interval? What, then, is the average slope of the graph relating y to x between these points?

2. If $y = 4x^3 - 19x + 10$, what is the angle in degrees between the x axis and the secant line to the graph of y with respect to x at the points where $x = 0$ and $x = 3$?

3. What is the slope of the straight-line graph relating y to x if $y = x$?

4. In Prob. 3, what is the angle in degrees between the graph and the positive portion of the x axis?

5. The current in a circuit was $i = 0.5t^2$. Using trigonometric tables, find at what angle a straight line is inclined which cuts the graph of this current where $t = 1$ second and where $t = 4$ seconds.

3-6 Average speed. Let an object move a varying distance s along a straight line, and let s be a function of time t. To determine the average speed $\mathbf{v}_{av}$ of this object over a given time interval Δt, we use this definition:

➢ The *average speed* of an object moving along a straight line is the average rate of change of distance with respect to time.

➢
$$\mathbf{v}_{av} = \frac{\Delta s}{\Delta t} \tag{14}$$

Do not confuse the meaning of s. This symbol stands for *distance.* It may be thought of as the space traveled. Note especially that s does *not* represent speed.

Speed $\mathbf{v}$ is expressed in such units as feet per second, miles per hour, or meters per second. As an example of an average speed calculation, suppose that an automobile moves 420 miles in a time interval of 10 hours. Then $\Delta s = 420$, and $\Delta t = 10$. Substituting these values in (14), we get $\mathbf{v}_{av} = 42$ miles per hour.

In conversation, *speed* and *velocity* are often used as if they had the same meaning. Strictly, there is a difference between them. The speed of an object is simply its rate of change of distance traveled, while its velocity includes not only this information but an indication of the object's *direction* as well. We might say that at a certain instant a train had a speed of 70 miles per hour. Its velocity, however, would properly include the direction, too, so that we could say that the train had a velocity of 70 miles per hour *toward the north.* Since velocity has both magnitude (size) and direction, it is a (space) vector quantity. Speed is only the magnitude of the quantity we call velocity. The symbol $\mathbf{v}$ is used to indicate velocity or its magnitude, speed.

QUESTIONS

1. Define the *average speed* of an object.
2. What is the meaning of the symbol s?
3. What is the difference in meaning between the terms *speed* and *velocity?*
4. What symbol represents speed? Velocity?

PROBLEMS

1. At a certain instant an object had moved a distance of 150 meters from its starting point; 3 seconds later it was 300 meters from the starting point. Find its average speed during the 3-second interval, assuming that its motion was along a straight line.

2. An electron moved a distance of 0.1 meter in a time 10^{-6} second. What was its average speed in meters per second?

3. The light-emitting spot on an oscilloscope screen was deflected over a straight path 8 centimeters long by a 1,000-cycle-per-second sine wave. What was its average speed during a one-way trip?

4. A freely falling object moves a distance s meters according to $s = gt^2/2$, where $g = 9.81$. What formula gives $\Delta s/\Delta t$? Find the average speed of fall from $t = 1$ to $t = 2$; from $t = 0$ to $t = 4$; from $t = 1$ to $t = 4$.

5. Points A, B, and C are located at 50-mile intervals along a straight highway. *a.* An automobile travels from A to B in 1 hour, then from B to C in 2 hours. Find its average speed. *b.* If the automobile moved from A to B in 1 hour, and then returned to A in 1 hour, what, strictly speaking, would be its average speed?

6. A projectile fired vertically upward had a height (h meters) at any instant given by $h = 800t - gt^2/2$, where $g = 9.81$. Find a formula for the average speed. Find v_{av} from $t = 10$ to $t = 11$.

7. An electron starting from rest in a uniform electric field moves according to $s = e\mathrm{E}t^2/2m_e$, where s is the distance in meters, t is the time of travel in seconds, e is the charge on the electron ($= 1.602 \times 10^{-19}$ coulomb), m_e is the mass of the electron ($= 9.106 \times 10^{-31}$ kilogram), and E is the electric-field intensity. Obtain the average speed of the electron over an interval of 0.1 microsecond, starting 0.1 microsecond after the electric field $\mathrm{E} = 50$ volts per meter was applied.

3-7 Looking forward. In this and in the preceding chapter, we have developed the following principal ideas: (*a*) that of a *function* of a variable and (*b*) that of the *average rate of change* of the function.

Our next major step will be to develop the idea of an *exact or instantaneous rate of change*, or, in graphical terms, to find the slope of the graph of a function at a single point, rather than the average slope over an interval. With relation to the *speed* problem, this idea allows us to find the instantaneous speed of an object, rather than merely its average speed over an interval of time.

This next step follows logically from what has already been done. It is necessary, though, that we first study the idea of a *limit* approached by a variable. This we shall do in the following chapter (see Fig. 3-9). We shall then be able to find the limit approached by the average rate of change, and this limit we shall define as the exact rate.

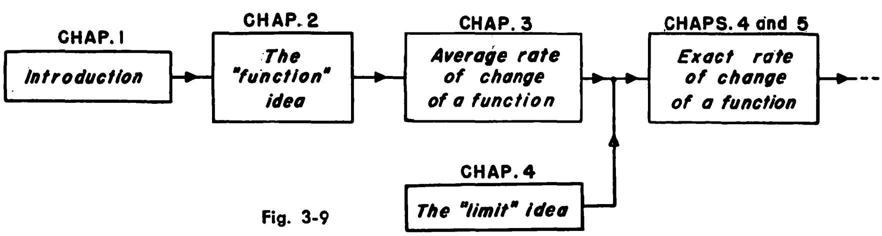


Fig. 3-9

4

Limits

In Chap. 3, we established the idea of an average rate of change. We turn our attention now to the problem of defining an exact rate of change. The idea of *limits* will be brought in to define exact rates and for other purposes.

4-1 Division by zero prohibited. The process of dividing a quantity a by a quantity b is defined as the finding of a quantity c, such that $bc = a$. Thus, when we write

$$10 \div 5 = 2$$

the meaning is precisely the same as if we had written

$$5 \times 2 = 10$$

It is, in general, impossible to divide a quantity by zero. To establish this fact, suppose that some quantity a could be divided by zero, getting a quotient c; that is, imagine that

$$a \div 0 = c \tag{1}$$

But if this statement were true, it would be exactly the same as saying that

$$a = 0 \times c$$

But when c (or any other quantity) is multiplied by zero, the product is zero, so that the latter equation is correct only if $a = 0$. And even if $a = 0$, then *any* value of c would satisfy (1), so that such an answer would be entirely useless.

Consequently, *the rules of mathematics prohibit any division by zero.* In other words, no meaning can be attached to a fraction having zero as its denominator, and we say that such a fraction is undefined. Mathematical rules in general do not apply when they result in a division by zero.

Note that nothing prevents our attaching a meaning to a fraction whose *numerator* is equal to zero. In fact, such a fraction has a very definite value, namely, zero. But if a fraction has a denominator equal to zero, then the fraction has *no value at all*—neither zero nor any other number.

Since a fraction having a zero denominator has no meaning, we can attach no meaning to the result of adding an actual number to such a fraction. Thus, $100 + 10/0$ is meaningless. Similarly, subtraction, multiplication, or division involving such undefined fractions gives meaningless results. We say that all such quantities are undefined.

PROBLEMS

State which of the following quantities have actual values, and give the values where they exist.

1. $12 - 0$
2. $0/7$
3. $5 \times 0/2$
4. $10 + 6/0$
5. $7/0$
6. $2 \times 5/0$
7. $0 \times 0/0$
8. $0 \div 3/2$
9. $7 + 3/0$
10. $5 \div 0/2$
11. $I = V/R$ if $R = 0$
12. $P = I^2R$ if $I = 0$
13. $X_L = 2\pi fL$ if $L = 0$
14. $X_C = 1/2\pi fC$ if $f = 0$
15. $Q = X_L/R$ if $R = 0$
16. $f_r = 1/2\pi\sqrt{LC}$ if $L = 0$

4-2 The limit idea. To assist in our work with exact or instantaneous rates, and for other purposes, we introduce the following definition:

> If a variable u approaches a constant c in such a way that the numerical value of $u - c$ becomes and remains less than any preassigned positive quantity, however small, then we say that u approaches c as a limit (written $\lim u = c$, or $u \to c$).

The idea of a limit, expressed in this definition, is basically important in all that follows in this book, both with regard to calculus and with regard to its applications to electricity. The definition should be remembered, and each of the following examples should be carefully studied so that the significance of the term *limit* becomes evident.

Example 1. In Fig. 4-1, a charged particle Q is located at a distance of 1 unit from a point P. Let Q approach P in a series of movements, each time traveling one-half the remaining distance separating it from P. Does this movement cause the distance PQ separating P and Q to approach a limit?

Let us make a table of the various distances separating P and Q, showing the progress of Q toward P. The first entry (Table 4-1) shows PQ to be 1 unit, representing the situation before any movement of Q takes place. After the first movement, the distance is reduced to $\frac{1}{2}$. The second movement reduces PQ to $\frac{1}{4}$, and so on. Inspection of Table 4-1 shows that the successive distances become smaller and smaller, approaching zero. In fact, we can make the difference between PQ and the constant *zero* as small as we please, simply by allowing Q to execute a sufficient number of movements according to its established law.

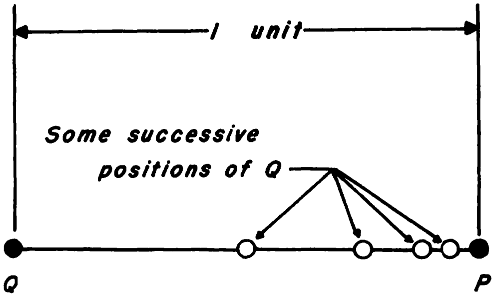


Fig. 4-1

We say, then, that *PQ approaches zero as a limit,* as the number of the movements n increases. That is, $\lim PQ = 0$, or $PQ \to 0$. (A variable which approaches zero as a limit is often called an *infinitesimal,* so that we could say that PQ is an infinitesimal.)

Table 4-1

Movement number, n	...	1	2	3	4
Distance PQ after the movement	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$, etc.

Example 2. Let a constant dc voltage be applied at time $t = 0$ to a circuit consisting of an inductance L in series with a resistance R. It may be shown that the resulting current i, instead of rising instantly to a value $i = V/R$, as would be indicated by Ohm's law, actually increases gradually toward this value. In fact, during each second of time i increases by a fixed fraction F of the remaining difference which separates it from the so-called final value V/R. If L and R are so proportioned that $F = 0.9$, show that the current approaches a limit as time progresses.

To show this, let us compute the value of i in terms of its final value V/R at the end of each successive second. The results are given in Table 4-2. It seems clear that, if we leave the circuit closed for a sufficiently long time, we can make

Table 4-2

Time, t seconds	0	1	2	3
i	0	$0.9V/R$	$0.99V/R$	$0.999V/R$, etc.

the difference between i and the constant V/R as small as we please—smaller than any value we might name. Then, according to our definition, $i \to V/R$ as time goes on.

If an independent variable x approaches some value k, and if, as a result, a function y of x approaches a constant c as a limit, we write $y \to c$ as $x \to k$, or

$$\lim_{x \to k} y = c \tag{2}$$

The latter is read "the limit of y, as x approaches k, is equal to c."* An instance of such a situation is given in the example below. It is essential to notice that, in finding such a limit, we are not trying to find what y is equal to when x is equal to k. We are finding what is often a very different quantity: the limit c which y approaches as x becomes very close to k.

Example 3. Let it be given that $y = (x^3 - 8)/(x - 2)$, and let it be desired to find $\lim\limits_{x \to 2} y$.

Substituting into this equation successive values of x closer and closer to 2, we get the results shown in Table 4-3. You should check one or two of these

Table 4-3

x..........	1	1.5	1.9	1.99	1.999
y..........	7	9.25	11.41	11.94	11.994, etc.

values to observe how they are calculated. Inspecting the table, we see that the difference between y and the constant 12 becomes smaller and smaller as x approaches 2. We say that y approaches 12 as a limit as x approaches 2, or

$$\lim_{x \to 2} y = 12$$

Before leaving this result, let us note that it could not have been obtained simply by substituting $x = 2$ in the given equation for y. For this would have resulted in

$$y = \frac{2^3 - 8}{2 - 2} = \frac{0}{0}$$

which is meaningless. That is, y *has no value* determined by the given formula when x is exactly equal to 2.

Example 3 points up the difference between merely *substituting* a value of x into an equation, as in elementary algebra, and *finding a limit* as x approaches some value.

* Another form is $\mathop{\mathrm{L}}\limits_{x \to k} y = c$.

PROBLEMS

1. Write in mathematical symbols:

(*a*) y approaches c as a limit.

(*b*) i approaches 5 as a limit.

(*c*) The limit of p, as r approaches 10, is 2.

(*d*) The limit of i, as t approaches 0.01, is 12.

(*e*) The limit of i_b, as v_c approaches -5, is 0.1.

2. If $y = (x^2 - 4)/(x - 2)$, show by a table of values what limit is approached by y as x approaches 2.

3. If $y = (x^2 - 9)/(x - 3)$, find by a table of values $\lim y$ as $x \to 3$.

4. In a dc circuit having a resistance of 10 ohms, what limit is approached by the power p as the current i approaches 1 ampere? Show tabulated values.

5. If a dc voltage of 20 volts is applied to a variable resistor, find by a table of values what limit the power in the resistor approaches as the resistance approaches 5 ohms.

6. What limit is approached by the reactance X_C of a 2-microfarad capacitor as the applied frequency approaches 20 cycles per second? Make a table of values.

7. A 1-volt signal is applied to a transmission line. If, over each mile of line, the signal loses 10 per cent of the voltage it had at the beginning of that mile, show by a table of values how the voltage v changes as a function of distance s. What limit is approached by v?

8. If, over a certain time, the current in a circuit is $i = (t^2 - t)/(t - 1)$, what limit does i approach as t approaches 1? Illustrate by means of a table.

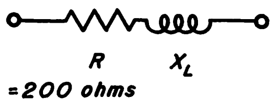


Fig. 4-2

9. What limit is approached by the absolute value of the impedance of a circuit like that of Fig. 4-2 as X_L approaches 150 ohms? Construct a table of values.

10. A circuit consists of a capacitance C in series with a resistance R. At time $t = 0$ this combination is connected to a dc source of 100 volts. It may be shown that, for this circuit, the voltage v_C across the capacitor rises, during each second of time after $t = 0$, by an amount equal to 80 per cent of the difference between v_C at the beginning of the second and 100 volts. Construct a table of values of v_C at various times, showing how it approaches a limit. What is the value of the limit?

11. Demonstrate by a table of values that the current through R in the bridge circuit of Fig. 4-3 approaches zero as R_4 approaches 100 ohms.

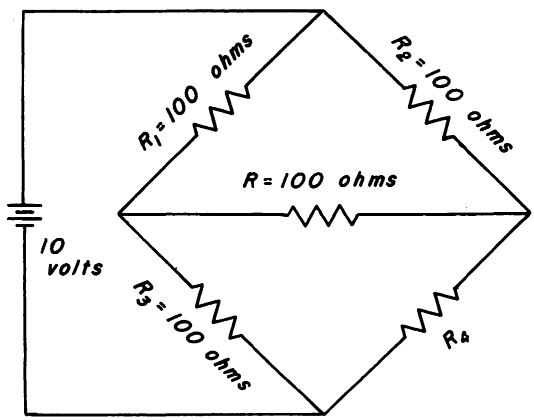


Fig. 4-3

4-3 Facts about limits. We shall find the following statements useful in our future work with limits:

1. If two equal variables approach limits, their limits are equal.
2. If two variables approach limits, the limit of their sum is the sum of their limits.
3. If two variables approach limits, the limit of their product is the product of their limits.
4. If two variables approach limits, the limit of their quotient is the quotient of their limits (see also Sec. 4-9).

These rules are quite simple, but they should be carefully noted. In certain courses, they are stated more formally and proved as theorems (see, for example, the chapter on "Limits" in ref. 1 at the end of this chapter). We shall depend upon actual use of these rules to bring home their significance to you as you progress through this book.

QUESTIONS

1. Write a definition of the term *limit.*
2. State four rules governing the application of limits.

4-4 Instantaneous speed. Exactly what do we mean when we refer to the speed *at a given instant* of a moving object? Nowhere in your studies preliminary to calculus have you learned a definition of the term *instantaneous speed.*

A suitable definition for instantaneous speed should provide an indication of the rate, in units of distance for each unit of time, at which the object is moving at a single instant. Clearly, the expression instantaneous speed does not mean the average speed of the object during the following hour, or even during the following minute or second. Yet, if we consider an average speed over a *very small* interval of time, we can get a value of average speed which is close to the instantaneous speed which we are thinking of. In fact, if we take smaller and smaller intervals of time, we can get values for the average speed which are *as close as we please* to the instantaneous speed. It is this idea which is used to form a precise definition of instantaneous speed:

➤ The speed **v** at any instant of an object moving in a straight line is defined as the limit approached by its average speed over an interval of time Δt as Δt is made to approach zero while always including the instant in question.

In symbols

➤ $$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \tag{3}$$

As an application of this definition, consider the luminous spot on the screen of a cathode-ray tube. Over a portion of its path, suppose that the spot moves a horizontal distance s according to

$$s = 10t - t^2 \qquad \text{centimeters} \tag{4}$$

where t is in seconds. Let it be required to find (a) a formula for the instantaneous speed $\mathbf{v}$ at any time t and (b) the value of the instantaneous speed when $t = 1$ second.

First, we shall get a formula for the *average* speed of the spot. [The formula which we shall eventually get for the instantaneous speed will be correct, in general, for any value of t for which Formula (4) applies, so that we shall no longer use the notation t_1 to indicate a particular point on the graph of s with respect to t.] The work proceeds as follows:

$$\begin{aligned} s + \Delta s &= 10(t + \Delta t) - (t + \Delta t)^2 \\ &= 10t + 10\Delta t - t^2 - 2t\,\Delta t - \Delta t^2 \end{aligned} \tag{5}$$

Subtracting (4) from (5),

$$\Delta s = 10\Delta t - 2t\,\Delta t - \Delta t^2$$

Dividing by Δt,

$$\frac{\Delta s}{\Delta t} = 10 - 2t - \Delta t \tag{6}$$

Equation (6) is a formula for the average speed of the spot over an interval Δt beginning at any time t.

We next *take limits* in (6). That is, we write a *new equation*, based upon (6), but instead of the quantities in (6), let us write the values of the limits approached by these quantities when Δt approaches zero [according to statement 1 of Sec. 4-3, these limits of the two members of (6) are equal].

In accordance with the definition (3), the quantity $\Delta s/\Delta t$ has as its limit the instantaneous speed $\mathbf{v}$ at time t. In the right member of (6) the quantity 10 is a constant, so that its value is not affected by changes in Δt. The quantity $-2t$ does not involve Δt and is therefore unchanged as Δt approaches zero. The final term, $-\Delta t$, has zero as its limit. Statement 2 of Sec. 4-3 establishes that the limit of the right member is the sum of the limits of 10, $-2t$, and $-\Delta t$. Our new equation relating the limits of the quantities in (6) is then

$$\mathbf{v} = 10 - 2t \tag{7}$$

which answers part (a) of our problem.

The answer to part (b) is found by substituting $t = 1$ in (7), giving

$$\mathbf{v}\ (\text{when } t = 1) = 10 - 2(1) = 8 \text{ centimeters per second}$$

Example. Neglecting air friction and other variables, an object falls in a time t seconds a distance given approximately by

$$s = 16t^2 \qquad \text{feet}$$

Find the instantaneous speed when $t = 1$ second.

The average is found as follows:

$$s + \Delta s = 16(t + \Delta t)^2 = 16t^2 + 32t\Delta t + 16\Delta t^2$$
$$\Delta s = 32t\Delta t + 16\Delta t^2$$
$$\frac{\Delta s}{\Delta t} = 32t + 16\Delta t$$

Taking limits, we write a new formula

$$\text{v} = 32t \qquad \text{feet per second} \tag{8}$$

Letting $t = 1$ second, we find that after 1 second of fall the object is traveling at 32 feet per second.

4-5 The derivative of a function. In the previous section, we defined a particular kind of exact rate of change. This was the instantaneous speed of an object. We now define the instantaneous or exact rate of change of a general function y with respect to a variable x; in other words, the rate of change of y at a *single point* on the graph relating y to x.

➤ This exact rate is defined as the limit approached by the average rate of change $\Delta y/\Delta x$ of y with respect to x as the interval Δx is made smaller and smaller while always including the point in question.

We call this exact rate the *derivative of y with respect to x* and indicate it by the symbol dy/dx. Thus

➤
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{9}$$

The value taken on by dy/dx when x has some particular value (say a) may be indicated by

$$\left(\frac{dy}{dx}\right)_{x=a} \qquad \text{or simply} \qquad \left(\frac{dy}{dx}\right)_a$$

Accordingly, the instantaneous speed of an object is actually the derivative of the distance s which it travels with respect to time t:

➤
$$\text{v} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \tag{10}$$

These definitions and symbols must be carefully learned. For the present, we shall not consider the symbol dy/dx to represent a fraction but simply the value of the exact rate described above.

The process of finding a derivative is called *differentiation*. One method of differentiation (called the *delta method*) was used in getting instantaneous speeds in the preceding section. To illustrate this method in the case of a function y, let it be given that

$$y = x^2 + 2x - 2 \tag{11}$$

and let it be desired to find a formula for dy/dx. We begin by getting a formula for $\Delta y/\Delta x$:

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 + 2(x + \Delta x) - 2 \\ &= x^2 + 2x\,\Delta x + \Delta x^2 + 2x + 2\Delta x - 2 \\ \Delta y &= 2x\,\Delta x + \Delta x^2 + 2\Delta x \\ \frac{\Delta y}{\Delta x} &= 2x + \Delta x + 2 \end{aligned} \tag{12}$$

Next we take limits in (12) as $\Delta x \to 0$. In accordance with the definition (9), the left member in the resulting new equation is dy/dx. The right member approaches the limit $2x + 2$. Then

$$\frac{dy}{dx} = 2x + 2 \tag{13}$$

This is a formula which gives the exact rate at which y is changing with respect to x, at any point x, expressed in units of y per unit of x. If, for example, we want to get this rate at the point where $x = 3$, we can substitute 3 for x in (13), getting

$$\frac{dy}{dx} = 2(3) + 2 = 8 \text{ units of } y \text{ per unit of } x \tag{14}$$

The differentiation procedure used above is of importance in obtaining basic formulas in later chapters. It must be fully understood. We may summarize this procedure as follows:

1. In the given equation for the function y, let the independent variable x undergo a change Δx, and write an equation for the resulting value of $y + \Delta y$.
2. Subtract the given equation for y from the equation for $y + \Delta y$, getting an equation for Δy.
3. Divide each term of the equation for Δy by Δx. This gives an equation for $\Delta y/\Delta x$.
4. Letting Δx approach zero as a limit, take limits in the equation for $\Delta y/\Delta x$. That is, write a new equation, relating the limits approached by the quantities in the equation for $\Delta y/\Delta x$. This new equation gives the desired derivative, dy/dx.

It must be kept in mind that dy/dx is a limit and is a quantity distinct from $\Delta y/\Delta x$. The equation for dy/dx is *not* simply another form of the equation for $\Delta y/\Delta x$: it is a new equation, obtained by observing what limits are approached by the quantities in the equation for $\Delta y/\Delta x$.

The derivative dy/dx expresses the amount by which y would change as x changed by 1 unit if y continued to vary at the rate at which it is changing at the point in question.

The derivative of a quantity with respect to time t is sometimes called its *fluxion.*

4-6 Some electrical applications of derivatives. The uses of derivatives, or exact rates, in electricity are numerous and important. Here we present some of the basic applications.

a. Instantaneous current. The instantaneous current i in amperes at any point in a circuit is the rate in coulombs per second at which electric charge q is being transmitted past the point at the instant under consideration. That is,

➣ The instantaneous current i is equal to the derivative of charge q with respect to time t,

or

➣ $$\boxed{i = \frac{dq}{dt}} \quad \text{amperes} \tag{15}$$

Example 1. The charge transferred past a certain point in a circuit varied according to $q = 5t^2 + 2t$ coulombs. Find a formula for the current i in the circuit at any time t.

Differentiating the equation for q according to the delta method,

$$q + \Delta q = 5t^2 + 10t\,\Delta t + 5\Delta t^2 + 2t + 2\Delta t$$

$$\Delta q = 10t\,\Delta t + 5\Delta t^2 + 2\Delta t$$

$$\frac{\Delta q}{\Delta t} = 10t + 5\Delta t + 2$$

$$\frac{dq}{dt} = i = 10t + 2 \text{ coulombs per second} = 10t + 2 \text{ amperes}$$

b. Induced voltage in a coil. In Fig. 4-4 we find a permanent magnet M which produces a field indicated graphically by the dotted lines of force. In the figure we have arranged the magnet so that its field is linked with

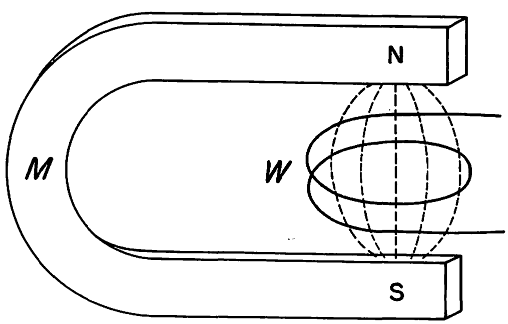


Fig. 4-4

the turns of a winding W. Let N represent the number of turns in the winding, and let ϕ represent the amount of magnetic flux in webers which actually passes through the turns of the winding (1 weber = 10^8 lines).

It has been observed experimentally that whenever we change the product $N\phi$ by any method, an emf v_{ind} will be induced in the winding (law of Henry and Faraday).

At present, let the number of turns be fixed, and suppose that the magnetic flux ϕ varies, as would be the case if the magnet were being withdrawn from the neighborhood of the coil. In such a case, it will be found that

➥ The induced emf at any instant is equal to the product of the number of turns N times the rate of change of flux ϕ which links the winding.

Using the derivative notation,

➥ $$\boxed{v_{ind} = -N\frac{d\phi}{dt}} \quad \text{volts} \tag{16}$$

(t is in seconds). The minus sign is in accordance with Lenz's law, conveying the fact that the current resulting from the induced emf is in such a direction as to oppose the change in flux.

Example 2. The magnetic flux through a 100-turn coil varied over a certain time interval in accordance with $\phi = 10t^3 + 5$ webers. Find the induced emf in the coil when $t = 0.1$ second.

Applying (16), we get $v_{ind} = -100\, d\phi/dt$. We find the value of $d\phi/dt$ by the delta method:

$$\begin{aligned}
\phi + \Delta\phi &= 10(t + \Delta t)^3 + 5 \\
&= 10t^3 + 30t^2\,\Delta t + 30t\,\Delta t^2 + 10\Delta t^3 + 5 \\
\Delta\phi &= 30t^2\,\Delta t + 30t\,\Delta t^2 + 10\Delta t^3 \\
\frac{\Delta\phi}{\Delta t} &= 30t^2 + 30t\,\Delta t + 10\Delta t^2 \\
\frac{d\phi}{dt} &= 30t^2 \qquad \text{webers per second}
\end{aligned}$$

so that $$v_{ind} = -100(30)(0.1)^2 = -30 \text{ volts}$$

c. Instantaneous power. The instantaneous power p in a circuit is defined as *the rate of change of energy* (or work) w at the instant in question. That is,

➥ $$p = \frac{dw}{dt} \qquad \text{watts} \tag{17}$$

where energy w is measured in joules and time t is in seconds.

Example 3. The energy dissipated by a resistor varied thus: $w = t + 6t^2$ joules. What was the power in the resistor when $t = 1$ second?

Differentiating the formula for w by the delta method,

$$\begin{aligned} w + \Delta w &= (t + \Delta t) + 6(t + \Delta t)^2 \\ &= t + \Delta t + 6t^2 + 12t\,\Delta t + 6\Delta t^2 \\ \Delta w &= \Delta t + 12t\,\Delta t + 6\Delta t^2 \\ \frac{\Delta w}{\Delta t} &= 1 + 12t + 6\Delta t \\ \frac{dw}{dt} &= p = 1 + 12t = 1 + 12(1) = 13 \text{ watts} \end{aligned}$$

QUESTIONS

1. Can any meaning be assigned to an *average speed* taken over a time interval of zero?
2. Define the term *instantaneous speed* of an object.
3. If y is a function of x, what do we mean by "the derivative of y with respect to x"?
4. What is the meaning of the term *differentiation?*
5. What symbol represents the *instantaneous speed* of an object?
6. What symbol represents the derivative of y with respect to x?
7. State the steps used in obtaining the derivative of y with respect to x by the delta process.
8. State in mathematical symbols the definition of *instantaneous current.*
9. Give an equation for the voltage induced in a winding at any instant when the magnetic flux through the coil is varying.
10. Define in mathematical terms the quantity *instantaneous power.*

PROBLEMS

Differentiate by the delta process in these problems.

1. $y = x^2 + x$
2. $y = 2x + 2$
3. $y = x^3$
4. $y = 3x^3 - 15$
5. $y = 6x - x^2$
6. $y = x^2 + x - 10$
7. $y = 5x^2 + 2x + 3$
8. $y = 4x^2 - 10x + 112$
9. $y = x^3 + 3x^2 - 21$
10. $y = 2x^3 - 5x^2 + 3x + 2$

11. A ball thrown vertically upward had a height h feet which varied according to $h = 40t - 16t^2$. What was its speed dh/dt when $t = 1$ second?

12. A projectile was fired upward so that its height h changed according to $h = 1{,}600t - 16t^2$. Find its rate of rise (a) when $t = 20$ seconds and (b) when $t = 30$ seconds.

13. A contact in a circuit breaker moved approximately according to the formula $s = 100{,}000t^3$, where s was in centimeters and t was in seconds. Find the speed of the contact when $t = 0.02$ second.

14. During a certain interval an electron in an electrostatic field moved a distance $s = 2.5 \times 10^{15}t^2$ meters, where t was in seconds. What was its speed after 0.001 microsecond?

15. If a capacitor is supplied with a charge q which varies according to $q = 2t^2$

coulombs, find (*a*) an equation for the current i flowing into the capacitor at any time t, (*b*) the current when $t = 0.05$ second, (*c*) the current when $t = 0.1$ second.

16. Same as Prob. 15, except let $q = t^2 - t^3/3 + 0.001$ coulombs.

17. The magnetic flux through a 500-turn winding varied, over a certain interval, according to $\phi = 0.004t$ webers. Find the induced emf in the winding (*a*) when $t = 0.01$ second and (*b*) when $t = 0.1$ second.

18. If the flux through a coil varied according to the formula $\phi = 0.01t - t^2 + 0.02$ webers, what voltage would be induced in the coil when $t = 0.02$ second? Assume that the coil has 150 turns.

19. If a winding is linked by a flux $\phi = 0.002t - 2t^2$ webers, how many turns must it have for the induced emf to be 8 volts when $t = 0.0025$ second?

20. The work done by a certain current varied with time according to $w = 6t^2 + 2t$ joules. Find the power in the circuit when $t = t_1$ seconds.

21. The energy supplied by a battery increased with time according to the formula $w = 3t + t^3$. Find the power in watts being delivered when $t = 1$ second.

4-7 Tangent lines; graphical treatment of derivatives. The curve of Fig. 4-5 depicts a variable y which is a function of x. Consider the secant line drawn through two points, A and B, on this curve. Now let point B move closer and closer to A, making the secant line take the

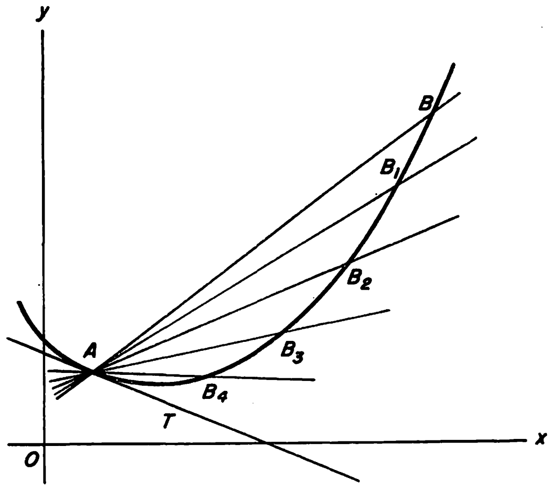


Fig. 4-5

successive positions AB_1, AB_2, AB_3, etc. Then as B draws nearer and nearer to A, the secant line approaches the limiting position indicated by line T. This line T we define as the tangent to the curve at point A.

➤ A tangent line to a curve at a given point is defined as the limiting position approached by a secant line through the given point as the second point at which the secant line cuts the curve approaches the given point.

This definition is attributed to the English genius Sir Isaac Newton (1642–1727), one of the two men credited with the invention of calculus. This definition[2,*] covers some cases which are not included by elementary definitions not using the *limit* idea. Another important definition is this:

➨ The slope of a curve at any point is defined as the slope of its tangent line at that point.

Figure 4-6 furnishes an illustration of the derivative of the function y with respect to x. The slope of the secant line, since it is a straight line,

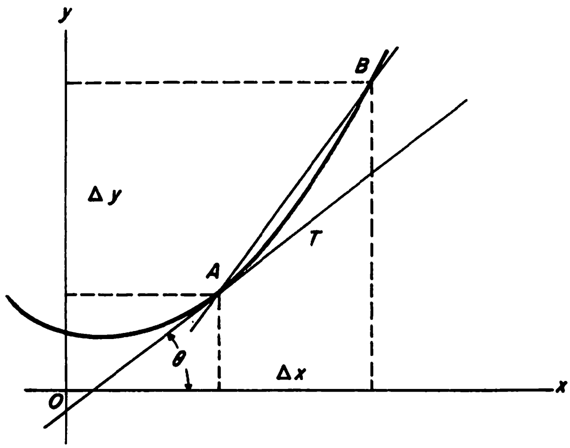


Fig. 4-6

has a constant value $\Delta y/\Delta x$ throughout its length. But $\Delta y/\Delta x$ is also the average rate of change of y with respect to x over the interval Δx. Now, if the interval Δx is made to approach zero, then the secant line will approach the tangent line as its limiting position. Thus, the slope of the tangent line is the limit approached by the slope $\Delta y/\Delta x$ of the secant line. But we have already defined this limit as the derivative dy/dx. We see, then, that

➨ The slope of the tangent line to a graph of y with respect to x, at a point A, is equal to the derivative of y with respect to x at point A.

This fact gives us a graphical method of finding the derivative of a function, as we shall demonstrate shortly. First, we note that

1. If the tangent line to a graph at a given point has a positive slope, then the graph must be rising as we proceed toward the right; that is, a

* These numerals refer to the list at the end of each chapter.

function y which is *growing* at a point on its graph has a *positive* derivative dy/dx at that point.

2. If the tangent line to a graph at a given point has a negative slope, then the graph must be falling as we proceed toward the right, so that a *negative* value of the derivative dy/dx is associated with a *decreasing* function y.

3. If the tangent to a graph at a given point is parallel to the x-axis, then the slope of the graph at that point is zero. Thus a zero derivative at a given point indicates that the function y is neither increasing nor decreasing at that point.

In Sec. 3-5, it was noted that a straight line has a constant slope along its entire length and that this slope has a value equal to the tangent of the angle between the line and the positive x axis. Applying this fact to the tangent line of Fig. 4-6, we see that

➠ While the derivative dy/dx is *defined* as the limit approached by $\Delta y/\Delta x$, it has a value given by the tangent of the angle θ between the tangent line and the positive x axis.

As an application of this fact, consider the grid-plate transconductance (mutual conductance) of a tube, which was defined in an elementary way in Chap. 3 as being equal to $\Delta i_b/\Delta v_c$. Clearly, the value obtained in this way at any point on a characteristic curve of a tube will depend upon how large an interval Δv_c is used in making the calculation. It is generally specified that a *small* interval should be used; but a more precise definition can be given for the mutual conductance of a tube at a given fixed plate voltage:

➠ The mutual conductance g_m of a tube at a given point on the graph relating its plate current i_b to its grid voltage v_c is equal to the rate of change of i_b with respect to v_c at that point.

➠ $$g_m = \frac{di_b}{dv_c} \qquad \text{mhos} \tag{18}$$

Accordingly, we can get the value of g_m for any given v_c simply by drawing a tangent line to the i_b-v_c graph at the point corresponding to the given v_c and measuring the slope of the tangent line. For example, the tube whose graph is shown in Fig. 4-7 has a g_m of 0.0026 mho (= 2,600 micromhos) when $v_c = -8$ volts. This value of g_m is obtained in either of two ways:

1. Draw the tangent line T to the i_b curve at the point where $v_c = -8$ volts. Note that in an interval from, say, -8 to -6 volts on the v_c axis, the tangent line rises by an amount 5.2 milliamperes on the i_b scale. Then $g_m = 0.0052/2 = 0.0026$ mho.

2. Alternatively, we can measure the angle between the tangent line T

and the v_c axis. This angle is about 45.5°. From a trigonometric table, tan 45.5° = 1.054. But each square on the i_b scale represents 0.001 ampere, while each square on the v_c scale represents 0.4 volt. To take into account this difference in scales, we multiply tan 45.5° (= 1.054) by 0.001/0.4 (= 0.0025), getting g_m = 0.0026 mho.

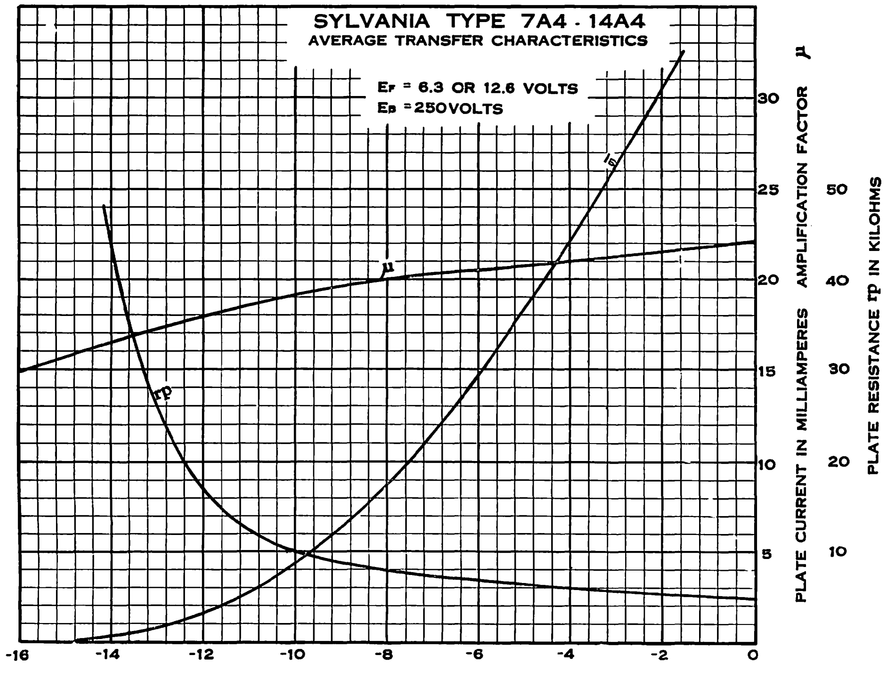


Fig. 4-7

Since the tangent line has a positive slope, g_m is positive, as we should expect.

This method, like graphical methods in general, is only approximate, but good results can often be had if care is used in laying off the tangent line. This line should fit *smoothly* along the curve. The portion of the tangent line on either side of the point of tangency should appear like a straight-line extension, with no sharp corner, of the curve from the other side of this point.

Other electron-tube characteristics may be defined in a similar manner:

$$r_p = \frac{dv_b}{di_b} \quad \text{ohms} \tag{19}$$

$$\mu = -\frac{dv_b}{dv_c} \tag{20}$$

QUESTIONS

1. Define the *tangent line* to a curve at a given point.
2. What is the meaning of the *slope of a curve* at a given point?
3. State the relation between the slope of the tangent line to a graph at a given point and the derivative of the graphed function at that point.
4. What fact is shown if the tangent line to a graph at a given point has a positive slope? A negative slope?
5. What is indicated if the tangent line to a graph at a given point is parallel with the horizontal axis?
6. State the relation between the derivative of y with respect to x at a given point on their graph and the tangent of the angle θ between the tangent line to the graph at the given point and the positive x axis.
7. Define, using calculus symbols, g_m, r_p, μ.

PROBLEMS

1. The distance s through which an object falls varies according to $s = 16t^2$ feet, where t was in seconds. Plot a graph of this formula from $t = 0$ to $t = 0.05$ second. Measuring the angle between the tangent line to this graph and the t axis and using trigonometric tables, find the speed when the object has fallen for 0.03 second.

2. Laying a straightedge tangent at the appropriate points to the graph of Fig. 4-7 estimate the values of g_m when (*a*) $v_c = -4$ volts and (*b*) $v_c = -10$ volts.

3. It was found that the plate voltage required to keep the plate current of a certain tube constant varied with grid voltage according to Table 4-4. (*a*) Plot a graph of v_b

Table 4-4

v_c..	0	−2	−4	−6	−8	−10	−12	−14	−16
v_b..	123	170	211	252	291	331	372	413	453

as a function of v_c. (*b*) Draw a tangent line to this graph at the point where $v_c = -2$ volts. (*c*) Measure the angle between this tangent line and the positive v_c axis. (*d*) Using tables of trigonometric functions, find the value of μ when $v_c = -2$ volts.

4. Similar to Prob. 3, except find μ when (*a*) $v_c = -8$ volts and (*b*) $v_c = -14$ volts.

5. At a given grid-bias voltage, the plate current of a tube had values for different plate voltages as shown in Table 4-5. (*a*) Plot these values on graph paper and

Table 4-5

v_b..................	130	150	200	250	300	350
i_b, milliamperes......	0	0.4	3.5	8.9	16.2	24.7

connect them with a smooth curve. (*b*) Using the tangent-line method, find the value of r_p according to (19) when $v_b = 200$ volts. (*c*) Similarly, find r_p when $v_b =$ 300 volts.

6. The flux ϕ in webers through a 165-turn winding varied according to the graph of Fig. 4-8. Find the induced emf when $t = 0.05$ second.

7. Same as Prob. 6, except let $t = 0.15$ second.

8. A resistor dissipated energy according to $w = t^2 - t + 1$ joules. Plot a graph from $t = 1$ to $t = 3$ seconds, draw a tangent line, and find the power in the resistor when $t = 2$ seconds.

9. A motor performed work according to $w = t^2 + 1$ joules. What power was being delivered by the motor when $t = 2$ seconds? Solve by plotting a graph and finding the slope of the tangent line.

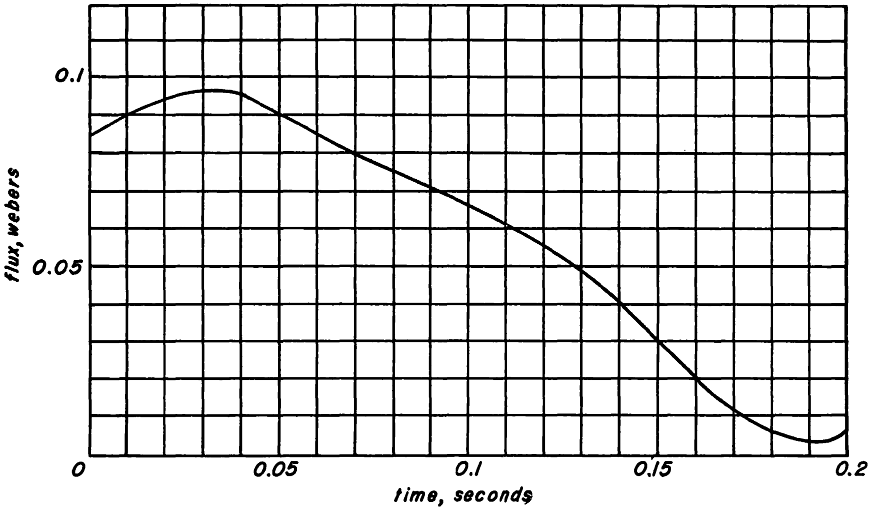


Fig. 4-8

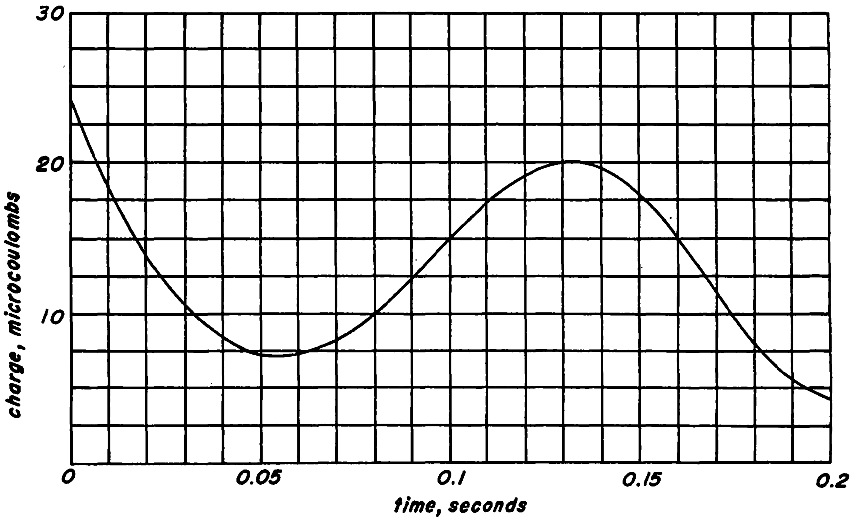


Fig. 4-9

10. Figure 4-9 shows the way in which a charge built up in a capacitor. Find the current in the capacitor when $t = 0.1$ second.

11. An electron influenced by an electric field moves in t seconds a distance given by $s = e\mathbf{E}t^2/2m_e$ meters. If $e = 1.602 \times 10^{-19}$ coulomb, $m_e = 9.106 \times 10^{-31}$ kilogram, and $\mathbf{E} = 10$ volts per meter, plot a graph of s from $t = 0$ to $t = 0.2$ microsecond.

Draw a tangent line to this graph and find the speed of the electron when $t = 0.1$ microsecond.

12. A roadway climbed so that its height followed the formula $h = 2{,}200 + 0.0002s^2$ feet, where s was horizontal distance. Plot a graph of h from $s = 0$ to $s = 100$ feet. Find, by drawing a tangent line, the *grade* of the roadway where $s = 50$ feet. [HINTS: (1) The grade of a roadway is given as a percentage and equals the number of feet to rise per 100 feet of horizontal distance. It may be found by moving the decimal point two places to the right in the value of the slope. (2) For this problem, best results are had by using a larger scale vertically than horizontally, letting 1 inch, for instance, represent 10 feet horizontally and 1 foot vertically. The resulting value of slope would then, of course, have to be divided by 10. (3) Let the s axis indicate $h = 2{,}200$ feet.]

13. An antenna wire had a height h which varied with distance x from its center point according to $h = 150 + 0.01x^2$ feet. Find its slope graphically where $x = 40$ feet [see hint (3) of Prob. 12].

4-8 Variables increasing without limit; infinity.

a. When a function increases without limit. Consider the function

$$y = \frac{1}{2 - x} \tag{21}$$

Let us calculate several values of the function y (Table 4-6) and plot a

Table 4-6

x	-1	0	1	1.5	1.8	1.9
y	0.3333	0.5	1	2	5	10

graph of this function as in Fig. 4-10. We note that y takes greater and greater values as x grows from -1 toward $+2$. It is seen that we could make y assume really tremendous values if we should make x almost, but not quite, equal to $+2$. For instance, if $x = 1.9999999$, we get $y = 10{,}000{,}000$. In fact, we can make y just as large as we please by letting x approach sufficiently close to $+2$.

As to what would happen if we should make x equal to $+2$, we see that this makes $y = 1/0$, which is meaningless. In other words, y has no value at all when $x = 2$.

We describe this behavior of y, in the neighborhood of the point where $x = 2$, by saying that

➼ y becomes *infinite* when $x = 2$.

In symbols,

➼
$$\lim_{x \to 2} y = \infty \tag{22}$$

It is important to avoid thinking that the function y in (21) *might* have a value when $x = 2$. There is *no* tremendous number *infinity* to which

y becomes equal when $x = 2$. We may properly say that "y becomes infinite when $x = 2$," with the understanding that this means

1. y has no value whatsoever when $x = 2$.
2. And y can be made as large as we please by making x sufficiently close to 2.

There exists a conflict in the use of the *limit* symbol in (22), since actually the function approaches no limit at all. But it is customary to use the form shown in (22), with the meaning just stated.

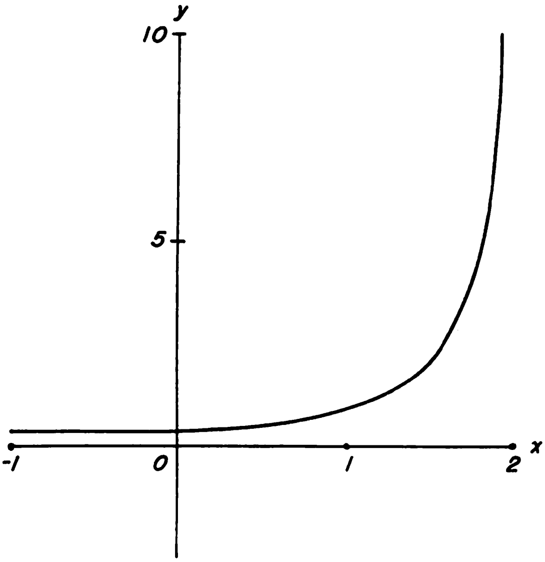


Fig. 4-10

Had y taken greater and greater *negative* values, rather than positive ones, as $x \to 2$, we should have said that "y becomes infinite negatively when $x = 2$" ($\lim_{x \to 2} y = -\infty$).

Example 1. In Fig. 4-11, the ratio of the length a to the length b is the tangent of the angle θ; that is, $\tan \theta = a/b$. Interpret from this figure the meaning of $\tan 90°$.

As θ is made to approach 90° through values smaller than 90°, a retains its constant value 1, while b becomes smaller and smaller. As $b \to 0$, $\tan \theta$ becomes enormously large. But when $\theta = 90°$, $\tan \theta$ takes the form $1/0$, which is undefined, so that $\tan 90°$ is a meaningless symbol. To indicate these facts, it is often written that $\tan 90° = \infty$, and in fact this is the symbol often shown in tables of functions. We should observe that this means (a) there is no such quantity as $\tan 90°$ and (b) $\tan \theta$ increases without limit as $\theta \to 90°$. (NOTE: This result depends upon our letting θ approach 90° through values of θ which are smaller than 90°. But in case θ decreases toward 90° from some larger value, then $\tan 90°$ becomes greater and greater *negatively*. With meanings for the symbols

as described above, it may be said that tan 90° is equal to either ∞ or $-\infty$. Similar results are obtained in many other cases.)

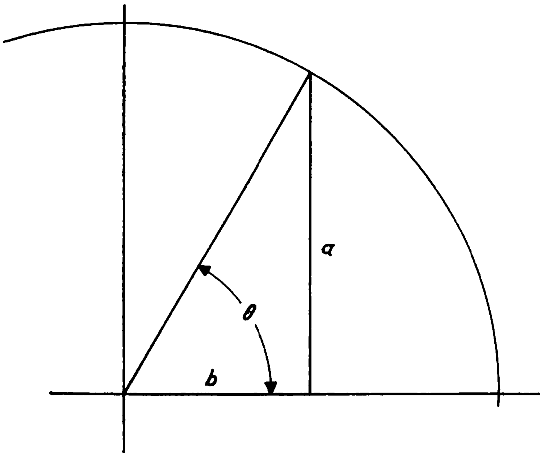


Fig. 4-11

b. When an independent variable increases without bound. Let it be given that

$$y = \frac{x}{2 + x} \tag{23}$$

and let it be desired to find the effect upon y of letting x increase without limit. That is, we wish to evaluate

$$\lim_{x \to \infty} \frac{x}{2 + x}$$

We might at first think that, in accordance with statement 4 of Sec. 4-3, we could take the quotient of the limits in the fraction of (23). But actually, neither numerator nor denominator has a limit, for both *increase without limit*, along with x. If, however, we divide both numerator and denominator by the greatest power of x which occurs in the denominator (in this case, simply the first power of x), we get

$$y = \frac{1}{2/x + 1} \tag{24}$$

Letting x now become infinite (increase without limit), we see that $2/x \to 0$. Therefore

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{1}{2/x + 1} = 1 \tag{25}$$

Another way of stating this result is to say that we can make y as close to 1 as we please simply by making x sufficiently large.

Example 2. Find

$$\lim_{x \to \infty} \frac{x^2 + 2x + 4}{x - 9}$$

Here again, both numerator and denominator become infinite along with x. If we divide numerator and denominator by x, however, we get

$$\frac{x^2 + 2x + 4}{x - 9} = \frac{x + 2 + 4/x}{1 - 9/x}$$

Here the denominator approaches the limit 1, but the numerator increases without limit as $x \to \infty$. Thus, the given fraction becomes infinite as x increases without bound, or

$$\lim_{x \to \infty} \frac{x^2 + 2x + 4}{x - 9} = \infty$$

A very important limit is

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

To estimate the value of this limit, let $(1 + 1/x)^x$ be called y, and let us tabulate some values of y taken as x is made larger and larger. For example, if $x = 100$, the calculations might go like this:

$$\begin{aligned} y &= (1 + 1/100)^{100} = (1.01)^{100} \\ \log_{10} y &= 100 \log_{10} 1.01 = 0.432 \\ y &= \text{antilog } 0.432 = 2.704 \end{aligned}$$

The way in which y changes as x grows without limit is shown in Table 4-7. (To obtain the last two entries as shown, it is necessary to use

Table 4-7

x.	1	2	10	100	1,000	10,000
y.	2	2.25	2.594	2.704	2.717	2.718

eight-place logarithm tables.) This table indicates that $(1 + 1/x)^x$ approaches a limit in the vicinity of 2.718 as x grows continually larger. In higher courses it is proved that this is true. The value of the limit is given approximately by $2.71828 \cdots$. (The series of dots indicates that the limit cannot be expressed exactly by a finite number of digits.) This number, referred to as e, occupies a prominent place in later sections of this book. You should remember that

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = 2.71828 \cdots \qquad \textbf{(26)}$$

In technical writings the limit e is sometimes indicated by the

symbol ϵ (Greek lower-case epsilon) or by ε, a special form of the letter e.

In a problem involving physical materials and equipment,

➢ A quantity may be considered infinite when any further increase in its value would produce no appreciable change in the results obtained.

For instance, a wave arriving from a distance of many miles has, for many purposes, only a negligible curvature of its wavefront, and we may therefore consider the wave as originating at an "infinite" distance, and take the wavefront as being "plane."

Example 3. In a certain vacuum-tube circuit, a gain A is obtained without feedback. When a fraction β of the output voltage is subtracted from the output circuit and fed back in phase with the input signal, it may be shown that the gain with feedback is

$$A_{fb} = \frac{A}{1 - A\beta}$$

Find the result of increasing β so that $A\beta \to 1$.

As $A\beta \to 1$, the denominator in the given fraction approaches zero. This results in enormous values of A_{fb}. If $A\beta$ is actually made equal to 1, the circuit has an "infinite" gain; that is, it provides an output signal even when no external input signal is applied. This output signal customarily is a sustained oscillation or *singing*. Actually the circuit then stops being useful as an amplifier and becomes an oscillator, so that an expression for its gain becomes meaningless.

QUESTIONS

1. What is the meaning of the statement that a certain function y "becomes infinite" for some value of the independent variable x?

2. What is the meaning of the entry $\tan 90° = \infty$ in a table of trigonometric functions?

3. What is the meaning of $\csc 180° = \infty$?

4. What would be the meaning of a statement that "a function u is equal to *minus infinity* when $x = 5$"?

5. What limit is represented by e? What is the approximate value of this limit?

6. In a system of physical materials and equipment, what is meant by a statement that a certain quantity "becomes infinite"?

7. A source delivers a constant voltage V, which is applied to a variable resistance r, so that the power in the resistance is $p = V^2/r$. Using a limit notation, like that of (22), indicate the effect of letting r approach zero.

8. It was stated that in a certain circuit "the current becomes infinite when the resistance is made equal to zero." What does this mean?

9. What would be the meaning of a statement that "the reactance of a capacitor becomes infinite at zero frequency"?

PROBLEMS

In Probs. 1 to 12 find the desired quantities, preparing tables or graphs where needful.

1. $\lim\limits_{x \to 1} \dfrac{3}{1 - x}$

2. $\lim\limits_{x \to -2} \dfrac{2}{x^2 - 4}$

3. $\lim\limits_{x \to 0} \dfrac{100}{x^2}$

4. $\lim\limits_{x \to 4} \dfrac{1}{x - 4}$

5. $\lim\limits_{x \to 3} \dfrac{25}{27 - x^3}$

6. $\lim\limits_{x \to -1} \dfrac{5}{1 + x}$

7. $\lim\limits_{x \to \infty} \dfrac{0.001}{x^2}$

8. $\lim\limits_{x \to \infty} \dfrac{7}{x}$

9. $\lim\limits_{x \to \infty} \left(x - \dfrac{1}{x}\right)$

10. $\lim\limits_{x \to \infty} \dfrac{5 + x^2}{1 - x^3}$

11. $\lim\limits_{x \to \infty} \dfrac{15x - 10}{x^2 + 5x + 10}$

12. $\lim\limits_{x \to \infty} \dfrac{x^3 + 2x^2 + 2}{x^2 + 9x + 1}$

13. Show by a table of values that $\lim\limits_{x \to \infty} (1 - 1/x)^{-x} = e$.

14. Same as Prob. 13, except use $\lim\limits_{x \to 0} (1 + x)^{1/x}$.

15. According to the theory of relativity, the mass m_a of a particle moving at a large velocity v past an observer appears to vary according to $m_a = m_0/\sqrt{1 - (\mathrm{v}/c)^2}$, where m_0 is the measured mass of the particle at rest and c is the speed of light ($= 3 \times 10^8$ meters per second). (*a*) Tabulate values of m_a for values of v nearer and nearer to c. (*b*) State in limit form the result of letting $\mathrm{v} \to c$.

16. In a feedback-amplifier problem discussed in Chap. 17, the gain of a certain type of amplifier with feedback is shown to be $A_{fb} = A(1 - \beta)/(1 - A\beta)$. If A is constant, and if β itself is allowed to take larger and larger values without bound, what happens to A_{fb}?

17. In studying feedback amplifiers the following formula is encountered: $(A\beta)_m = 2(N + 1)^2/N$. Find the form approached by $(A\beta)_m$ as $N \to \infty$.

4-9 Existence of limits. In reality, it is usually a difficult matter to find what limit is approached by a variable or even to prove conclusively whether or not the variable actually approaches a limit.[3]

It should be kept in mind, then, that whenever we speak of anything which is defined as a limit, we are assuming that this limit actually exists —which in reality it may or may not do. For instance, consider the limit approached by the quotient of two variables, both of which approach limits. It may be shown that, if the divisor approaches zero, the *quotient may or may not approach a limit.*[3]

The derivative of a function, for example, is defined as the limit approached by a quotient as the divisor approaches zero. But not all functions *have* derivatives at every point. A square wave of current, for example, supposedly has points at which it rises *vertically*, so that it

increases by a definite amount in a zero interval of time. At such a point, then, its slope or derivative is nonexistent. Likewise, a function which has *breaks* in its graph has no derivative in the intervals where the breaks appear. In general, for a function to have a derivative throughout a certain interval, it must be continuous within that interval, having neither breaks nor abrupt jumps. Requirements for the existence of derivatives are treated in higher courses.

Functions which occur in nature, including those of interest in the electronics field, have in general only a limited number of discontinuities, if any. It is even theoretically impossible to produce an absolutely square wave of current in a practical circuit, although the wave may be made to approach squareness very closely. For our purposes, we shall assume that the variables discussed have derivatives at all points of interest unless stated otherwise.

REFERENCES

1. H. L. Rietz and A. R. Crathorne: "College Algebra," 5th ed., Henry Holt and Company, Inc., New York, 1951.
2. F. L. Griffin: "Introduction to Mathematical Analysis," rev. ed., pp. 66–67, Houghton Mifflin Company, Boston, 1936.
3. H. M. Bacon: "Differential and Integral Calculus," 2d ed., pp. 11–18, McGraw-Hill Book Company, Inc., New York, 1955.

Part Two

BASIC OPERATIONS

5

Derivatives

We now prepare some *differentiation formulas*, by which exact rates can be calculated more rapidly and conveniently than by the delta process.

5-1 Derivative operator. To indicate the derivative of a function we may write the symbol d/dx, called the *derivative operator*, before the function. Consider the function

$$x^2 + 2x$$

for example. Its derivative may be indicated by

$$\frac{d}{dx}(x^2 + 2x)$$

If the function $x^2 + 2x$ is actually differentiated, as by the delta method, the derivative obtained is $2x + 2$. Then we may write

$$\frac{d}{dx}(x^2 + 2x) = 2x + 2$$

The symbol d/dx can be interpreted, then, as indicating "the derivative with respect to x of the function which follows." For instance, the derivative dy/dx could also be written

$$\frac{d}{dx}y$$

Differentiation rules will be stated both as sentences and as formulas. The rules should be remembered as statements in words.

5-2 Derivative of an isolated constant. Let y be a function of x such that y has a constant value a for all values of x, as shown in Fig. 5-1.

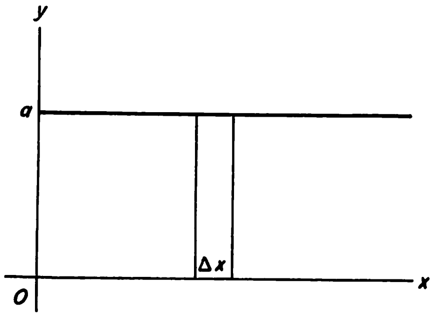


Fig. 5-1

If x takes some initial value, as shown, and if then x goes through a change Δx, then y makes no change at all. That is, $\Delta y = 0$, so that

$$\frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$$

To get dy/dx, we write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

Since $y =$ the constant a,

$$\frac{d}{dx} a = 0 \qquad (1)$$

a is always a constant

or

The derivative of any constant is equal to zero.

This result is consistent with that noted in Sec. 4-7.

Example. The current in a circuit is $i = 6$ amperes. Find di/dt.
By Formula (1),

$$\frac{di}{dt} = 0$$

5-3 Derivative of a variable with respect to itself. Suppose that $y = x$, as graphed in Fig. 5-2. If x takes some initial value and then goes through a change Δx, then y makes a change $\Delta y = \Delta x$. Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x}{\Delta x} = 1$$

and $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

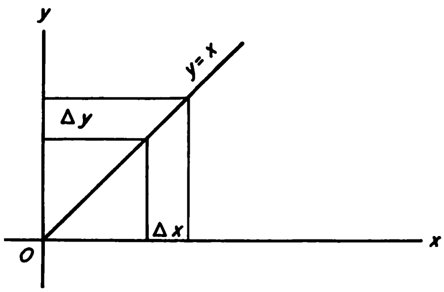


Fig. 5-2

Since $y = x$, this gives

$$\frac{d}{dx} x = 1 \qquad (2)$$

or

The derivative of a variable with respect to itself is equal to 1.

Example. The power p in a resistor varied with time t so that $p = t$. Find the rate of change of power with respect to time at any time t.

From (2), $dp/dt = 1$ watt per second.

5-4 Derivative of a power of a variable. Let $y = x^n$, and let it be desired to find an equation for dy/dx. This derivative is readily found in many cases by the delta method. Some examples follow.

CASE I. Let $y = x^2$. You can quickly show that $dy/dx = 2x$.

CASE II. Let $y = x^3$. Here you can easily show that $dy/dx = 3x^2$.

CASE III. Let $y = x^4$. Then

$$\begin{aligned}
y + \Delta y &= (x + \Delta x)^4 \\
&= x^4 + 4x^3\,\Delta x + 6x^2\,\Delta x^2 + 4x\,\Delta x^3 + \Delta x^4 \\
\Delta y &= 4x^3\,\Delta x + 6x^2\,\Delta x^2 + 4x\,\Delta x^3 + \Delta x^4 \\
\frac{\Delta y}{\Delta x} &= 4x^3 + 6x^2\,\Delta x + 4x\,\Delta x^2 + \Delta x^3 \\
\frac{dy}{dx} &= 4x^3
\end{aligned}$$

Notice that in each of these cases the power of x appearing in the derivative has an exponent which is *one less* than that of the power of x in the given function being differentiated. Also, the power of x in the derivative is multiplied by a constant equal to the original exponent. It will be shown in Chap. 12 that these facts apply to any other powers of x which we might differentiate, so that we have the following rule:

The derivative of a power of x is equal to the given exponent multiplied by x to a power one less.

Stated as a formula,

$$\frac{d}{dx}x^n = nx^{n-1} \tag{3}$$

Example 1. In a certain circuit the voltage was varied according to $v = t^5$. Find the rate of change of voltage when $t = 2$ seconds.

Applying Formula (3), we get $dv/dt = 5t^4$. Letting $t = 2$, we calculate $dv/dt = 5(2)^4 = 80$ volts per second.

Example 2. If $i = t^{-3/2}$, find di/dt.

Using (3),

$$\frac{di}{dt} = -\frac{3}{2}t^{-3/2-1} = -\frac{3}{2}t^{-5/2} \qquad \text{amperes per second}$$

5-5 Effect of a constant multiplier. Consider a quantity y which is a function of a variable x. If y is multiplied by some constant a, how will the derivative of this function be affected?

Figure 5-3 shows a graph of y. There is also shown a graph of a

second function Y, which is everywhere equal to y multiplied by some constant a (in the figure, a has been made equal to 2). That is,

$$Y = ay \tag{4}$$

Now, let x begin at some initial value and go through a change Δx, taking a new value $x + \Delta x$, as illustrated. Then Y changes from its initial value to a new value $Y + \Delta Y$. Similarly, y changes from its original value to the value $y + \Delta y$. But since the value of Y is always equal to a times y,

$$Y + \Delta Y = a(y + \Delta y) = ay + a\,\Delta y \tag{5}$$

Substituting (4) into (5),

$$ay + \Delta Y = ay + a\,\Delta y$$
$$\Delta Y = a\,\Delta y$$

a is always a constant

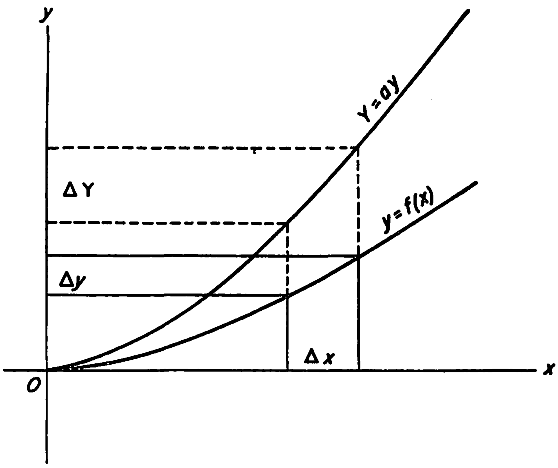


Fig. 5-3

Dividing by Δx and taking limits as $\Delta x \to 0$,

$$\frac{\Delta Y}{\Delta x} = a\,\frac{\Delta y}{\Delta x}$$
$$\frac{dY}{dx} = a\,\frac{dy}{dx}$$

Since $Y = ay$,

$$\frac{d}{dx}(ay) = a\,\frac{dy}{dx} \tag{6}$$

In words,

If a variable is multiplied by a constant a, then its derivative is also multiplied by a.

Notice especially that although the derivative of an *isolated* constant is zero, the effect of *multiplying* a variable by a constant is quite different.

As an application of this result, let us get a formula for the derivative of a function

$$y = ax^n \tag{7}$$

where a and n are constants. According to (3), the derivative of this function *if the constant a were neglected* would be simply nx^{n-1}. Applying (6), the effect of the multiplier a is to multiply the derivative by a, so that

$$\frac{d}{dx}(ax^n) = nax^{n-1} \tag{8}$$

We say that

The derivative of a power of a variable multiplied by a constant is equal to the product of the exponent of the original power times the constant multiplier times the variable to a power one less.

Example 1. If $y = 3x^5$, find dy/dx.
Applying (8), we see that $a = 3$ and $n = 5$, so that

$$\frac{dy}{dx} = (5)(3)(x^{5-1}) = 15x^4$$

Example 2. If $i = 2/t^3$ amperes, find di/dt.
We may write $i = 2t^{-3}$, so that

$$\frac{di}{dt} = (-3)(2)(t^{-3-1}) = -6t^{-4} \qquad \text{amperes per second}$$

Example 3. An object moves a distance s feet in a time t seconds according to $s = 100t^2$. Find its speed when $t = 2$ seconds.

By definition, the speed $\mathbf{v}$ is equal to ds/dt, so that

$$\mathbf{v} = (2)(100t) = 200t$$

Letting $t = 2$, we get $\mathbf{v} = 400$ feet per second.

QUESTIONS

1. Give the meaning of the symbol d/dx written before a function.
2. State in words the value of the derivative of any constant.
3. State in words the value of the derivative of a variable with respect to itself.
4. Give in words the rule for obtaining the derivative of a variable raised to a power.
5. If a function is multiplied by some constant, what is the effect upon its derivative?
6. Give the rule for obtaining the derivative of a constant multiplied by a power of a variable.

PROBLEMS

In Probs. 1 to 13 find dy/dx.

1. $y = 5$
2. $y = 1{,}000$
3. $y = x$
4. $y = ax$
5. $y = x^5$
6. $y = x^{-1.3}$
7. $y = 2x^7$
8. $y = ex^{\pi}$
9. $y = \sqrt{x}$
10. $y = 5\sqrt[5]{x}$
11. $y = 6x^{-15}$
12. $y = \pi/\sqrt[3]{x^4}$
13. $y = 5/(8\sqrt[8]{x^5})$

14. A charged particle moved a distance s in time t according to $s = kt^2$, where k was constant. Find its speed v at any time.

15. The charge in a capacitor varied according to $q = 1{,}000t^3$. What was the current i when $t = 0.001$ second?

16. The magnetic flux ϕ in a coil of 600 turns varied as $\phi = 0.3t^{5/3}$ webers, where t was in seconds. Find the induced voltage v_{ind} when $t = 1$ second.

17. The work w performed by an electromagnet varied, during a certain time interval, as $w = 27t^3$. What was the power p, in watts, delivered by the magnet when $t = 1/6$ second?

18. The low-frequency inductance of a coil of N turns is given as $L = FN^2d$, where F is a *form factor* and d is the diameter of the coil. Find dL/dN.

19. The energy stored in a capacitor is $w = Cv^2/2$. Find the rate at which the stored energy varies as the voltage is changed.

20. A storage cell was given a charge which varied according to $Q = 9t^{2/3}$ ampere-hours, where t was in hours. What was the charging current in amperes after 18 hours?

5-6 Some electrical applications.

Using rule (6), we are able to derive some important formulas for use in circuit work.

a. Current in a capacitor. According to an elementary formula, the charge q coulombs in a capacitor of constant capacitance C farads varies with the applied capacitor voltage v according to

$$q = Cv \qquad \text{coulombs} \tag{9}$$

Now let v be varied, so that q also changes. Since the two members of (9) are equal, their rates of change with respect to time t must also be equal. To find these rates we express the derivatives of both members of (9). The rate of change of q we may call dq/dt. The rate of change of the right member is

$$\frac{d}{dt}(Cv) = C\frac{dv}{dt}$$

by (6). Therefore $dq/dt = C\,dv/dt$. But dq/dt is the value of the current i_C in the capacitor at any instant (Sec. 4-6). Thus

$$i_C = C\frac{dv}{dt} \qquad \text{amperes} \tag{10}$$

Among the many consequences of this result, we note that the voltage across a capacitor *cannot* instantly change from one value to another.

For this would mean that dv/dt must be infinite, which would require an infinite current i_C according to (10), and an infinite current cannot exist in a physical sense. A square wave of voltage is a theoretical impossibility, for example, since every circuit must be assumed to have at least a little capacitance.

b. Induced voltage in an inductor.

The inductance L in henrys of a coil is defined as the product of the number of turns N times the magnetic flux ϕ in webers passing through the coil divided by the current i in amperes required to produce this flux.

That is,

$$L = \frac{N\phi}{i} \quad \text{henrys} \tag{11}$$

While this definition is perhaps not familiar to you, some calculus operations will relate it to your previous experience.

We may rewrite (11) as

$$Li = N\phi \qquad \frac{d}{dt}(Li) = \frac{d}{dt}N\phi \tag{12}$$

Assuming that L and N are constants, let the current i vary as required to produce any desired flux ϕ. We now differentiate (12). Taking into account the constant multipliers L and N gives

$$L\frac{di}{dt} = N\frac{d\phi}{dt} \qquad L\frac{di}{dt} = \frac{d\phi}{dt}$$

But the right member is the negative of the expression for the *voltage induced in a coil* (Sec. 4-6). Therefore we write

$$v_{ind} = -L\frac{di}{dt} \quad \text{volts} \tag{13}$$

It is doubtless confirmed by your experience that a coil produces a negative, or kickback, voltage proportional to the product of its inductance and the rate at which the current is changed.

Note that the current in a coil cannot be changed instantly from one value to another—for this would require an infinite voltage v_{ind} across the coil. Accordingly, a perfect square wave of current theoretically cannot be obtained, since every circuit must contain some inductance.

c. Mutual inductance. Let two coils be coupled together (Fig. 5-4) so that a magnetic flux ϕ_2 webers links winding 2 as a result of the current i_1 in winding 1. If winding 2 has N_2 turns, we define the *mutual inductance* between the windings as

$$M = \frac{N_2\phi_2}{i_1} \quad \text{henrys} \tag{14}$$

Rearranging and differentiating,

$$N_2 \frac{d\phi_2}{dt} = M \frac{di_1}{dt} \tag{15}$$

But the left member is, by Sec. 4-6, the negative of the expression for the voltage v_2 induced in winding 2. So we write

$$v_2 = -M \frac{di_1}{dt} \qquad \text{volts} \tag{16}$$

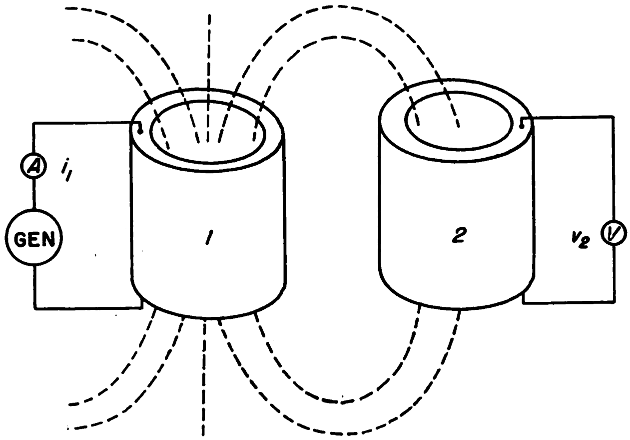


Fig. 5-4

The minus sign is a convention which satisfies Lenz's law. (The polarity of the "output" voltage v_2 can be reversed by reversing the connections to winding 2.)

QUESTIONS

1. State a formula for the current flowing into a capacitor at any time t.
2. Give a formula which defines the inductance of a coil.
3. What equation expresses the voltage induced in a coil in terms of its inductance and the rate of change of current?
4. Give a defining formula for the mutual inductance between two windings.
5. What equation gives the value of the induced voltage in a coil when the current is varied in a second coil coupled to it?

PROBLEMS

1. A voltage $v = 200t^2$ volts is applied to a 1-microfarad capacitor. What formula gives the current in the capacitor?

2. Find the current through a capacitor of 0.01 microfarad capacitance when $t = 0.01$ second if the applied voltage is $v = 100t^{3/2}$ volts.

3. If the current in a 2-microfarad capacitor is to be 3 milliamperes, at what rate in volts per second must the applied voltage be changed?

4. What formula gives the induced voltage in a 10-henry coil when the current is $5t^{1/2}$?

5. What formula expresses the voltage v_{ind} across a 100-millihenry coil if the current is $i = 0.2$ ampere?

6. How fast does the current in a 12-henry coil change to cause an induced emf of 3.6 volts?

7. The mutual inductance between two coils is $M = 6$ henrys. How fast must the current in one of the coils vary in amperes per second to induce -4.8 volts in the other coil?

8. If a current $i_1 = 13.5t^{2/3}$ amperes is transmitted through a coil, how much voltage v_2 is induced in a second coil when $t = 0.001$ second? The mutual inductance between the coils is 0.2 henry.

9. If the current in a 30-henry inductor is changed according to $i = 0.1t^3$, after what time will the induced voltage be -81 volts?

10. The voltage applied to a 10-microfarad capacitor varied with time (t seconds) according to $v = 2{,}500t^{4/3}$. After what time was the current equal to 10 milliamperes?

11. A current $i_1 = t^2$ is sent through a 9-henry coil. This coil is coupled to a second coil so that their coefficient of coupling is 0.5. If an emf of -3 volts is induced in the second coil when $t = 0.5$ second, what must be the inductance of the second coil?

5-7 Derivative of a sum.

Let

$$y = u + v \tag{17}$$

where u and v are functions of x. Let u and v go through changes Δu and Δv, respectively, as a result of a change Δx in x. Let the resulting change in y be called Δy. Then

$$y + \Delta y = u + \Delta u + v + \Delta v \tag{18}$$

Subtracting (17) from (18),

$$\Delta y = \Delta u + \Delta v$$

Dividing each term by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

Now we let Δx approach zero as a limit. The limit approached by the left member is dy/dx. By statement 2 of Sec. 4-3, the limit approached by the right member is the sum of the limits of the separate terms. These two terms approach the limits du/dx and dv/dx, respectively, so that

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Substituting (17) into this,

$$\boxed{\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}} \tag{19}$$

In words,

➢ The derivative of the sum of several functions is equal to the sum of the derivatives of these functions.

(Since this statement is true for two functions, it may be extended directly to include more than two functions.)

PROBLEMS

Know

In Probs. 1 to 15 differentiate with respect to x.

1. $y = x^2 + 5x$
2. $y = 2x + 7x^3$
3. $y = 10x^3 + 15x$
4. $y = 5x - 3x^5$
5. $y = x - x^2$
6. $y = x^8 + x + 10$
7. $y = x^6 + 20x + 16$
8. $y = x^2 + 21x + 5$
9. $y = 7x^5 - 20x^3 + 12$
10. $y = x^3 + 7x^2 + 2x + 11$
11. $y = \sqrt{x} + 12$
12. $y = x^{3/2} + \sqrt{x} + 10$
13. $y = x^{5/3} + 2x^{-3/2} + x$
14. $y = x^3 + x^{3/2} + 2x + 7$
15. $y = x^{2/3} - x^{-2/3} + \sqrt[3]{x} + 3$

16. A radio receiver parachuted to earth fell so that its height was $h = 800 - 18t$. What was the speed with which it struck the earth?

17. An object was dropped from an airplane so that its height (h feet) varied according to $h = 12{,}000 - 16t^2$. What was its speed after 10 seconds?

18. The emf v produced by a thermocouple varies with temperature T approximately as $v = A + BT + CT^2$, where A, B, and C are constants. Find the rate at which v varies with respect to T (called the *thermoelectric power* of the couple).

19. An antenna wire was hung so that its height h feet at any horizontal distance s feet from the center was $h = 70 + 0.0005s^2$. What was the slope of the wire where $s = 40$ feet?

5-8 Derivative of a function of a function. Let y be a function of a variable u, and let y change with u in the manner shown in Fig. 5-5a, for example. Now let u, in turn, be a function of still another variable x, as illustrated, for example, in Fig. 5-5b. Let it be desired to find a formula for the rate of change of y with respect to x.

We have here a case of *indirect* dependence of y upon x, as discussed in Sec. 2-9. We say that y is a function of x through the variable u.

If we let x take some initial value and then go through a change Δx, we may let Δu represent the resulting change in u (Fig. 5-5b). But while u goes through this change, y will make a change Δy (Fig. 5-5a). We may, of course, write an equation stating that $\Delta y/\Delta x$ is equal to itself, and we may also multiply both the numerator and the denominator of this fraction by Δu:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y\, \Delta u}{\Delta x\, \Delta u}$$

This may be written

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x} \tag{20}$$

If we now let Δx approach zero, Δu must also approach zero, as also must Δy. (Needless to say, these quantities do not necessarily approach zero in the same manner.) But the limit approached by the left member

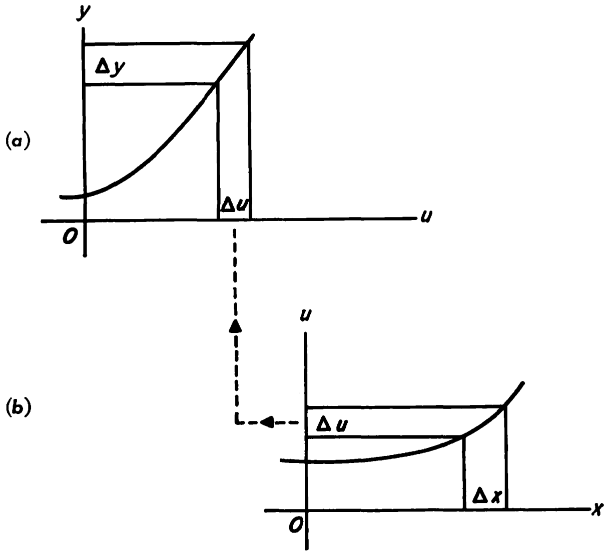


Fig. 5-5

of (20) then becomes dy/dx. The first factor in the right member approaches dy/du, while the second factor approaches du/dx, giving

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{21}$$

As an example, the power in a circuit varies with current according to

$$p = Ri^2 \tag{22}$$

Let the resistance R be a constant, and suppose that i varies with time according to

$$i = 5t^2 \tag{23}$$

To find dp/dt we first differentiate (22), getting

$$\frac{dp}{di} = 2Ri$$

Differentiating (23),

$$\frac{di}{dt} = 10t$$

Applying (21),

$$\frac{dp}{dt} = \frac{dp}{di}\frac{di}{dt} = 2Ri(10t) = 20Rit$$

This gives the desired derivative of p with respect to t. If desired, (23) can be substituted into this result, giving dp/dt in terms of t alone:

$$\frac{dp}{dt} = 20Rt(5t^2) = 100Rt^3$$

PROBLEMS

1. A resistance changes according to $r = t^3 + 5$ ohms. (*a*) Given that the voltage across the resistance is $v = Ir$, find a formula for dv/dt. (*b*) Find dv/dt when $t = 2$ seconds if $I = 10$ amperes.

2. The wavelength λ meters of a radio wave moving at a speed $c = 3 \times 10^8$ meters per second varies with frequency according to $\lambda = c/f$. If $f = 10^8 + (5 \times 10^7)t^{1/2}$ cycles per second, find a formula for $d\lambda/dt$.

3. The energy stored in the magnetic field of an inductor is $w = Li^2/2$ joules. (*a*) If $L = 12$ henrys, and if $i = 2 - 6t^2$, find a formula for dw/dt. (*b*) At what rate in watts is the stored energy being dissipated when $t = 0.5$ second?

4. The intensity I of the light from a tungsten filament varies with the applied voltage according to $I = AV^{3.7}$, where A is a constant and V is the applied voltage. If $V = t - 2t^2$, find a formula for dI/dt.

5. The force between two charged particles varies with the distance s separating them according to $\mathbf{F} = q_1q_2/4\pi\epsilon s^2$. If q_1, q_2, and ϵ are constant, and if s varies with time t as $s = 6t^{3/2}$, find a formula for $d\mathbf{F}/dt$.

6. The mutual inductance between two coils is $M = N_2\phi_2/i_1$, where i_1 is the current in one of the coils and N_2 and ϕ_2 are, respectively, the number of turns of the second coil and the flux linking it to the first coil. If i_1 and N_2 are constant, and if the second coil is moved so that ϕ_2 varies with time (t seconds) according to $\phi_2 = t^3 - 2t$, find a formula for dM/dt.

7. The resistance of a copper wire of length s feet is given by $R = 10.8s/d^2$, where d is the diameter in mils. If a sliding contact changes the length so that $s = t^2 + 0.5t$, at what rate, in ohms per second, is the resistance changing when $s = 1{,}000$? Let $d = 80$.

8. When a length l meters of a conductor moves at a speed v meters per second in a magnetic field of uniform flux density $\mathbf{B}$ teslas (webers per square meter), a voltage is induced equal to $v = -\mathbf{B}l\mathrm{v}$ volts. If $\mathrm{v} = 10$ meters per second, $l = 0.3$ meter, and $\mathbf{B}$ varies over a certain time interval according to $\mathbf{B} = 1/t^2$, find dv/dt when $t = 0.5$ second.

9. A certain factory producing radio instruments at a rate of u per month makes an expenditure A depending upon its production according to $A = 10{,}000 + 26u - 0.02u^2$. But the number produced (and sold) varies with the selling price x dollars for each instrument according to $u = 1{,}000 - 5x$. At what rate does the expenditure change with respect to the selling price when $x = 100$?

10. The frequency of a certain crystal oscillator varies with temperature T according to $f = f_a[1 + k(T - T_a)]$, where f_a is the frequency at an initial temperature T_a and k is a constant of the crystal. If T varies with time (t minutes) according to $T = 55 + 0.01t^2$, how fast is f changing when $t = 10$?

5-9 Derivative of a power of a function. Let it be desired to obtain the derivative

$$\frac{d}{dx}(au^n)$$

where u is some function of x and a and n are constants. According to Formula (8),

$$\frac{d}{du}(au^n) = nau^{n-1}$$

But by (21),

$$\frac{d}{dx}(au^n) = \frac{d}{du}(au^n)\frac{du}{dx}$$

or

$$\boxed{\frac{d}{dx}(au^n) = nau^{n-1}\frac{du}{dx}} \tag{24}$$

In words,

The derivative of a power of a function multiplied by a constant is equal to the product of the exponent of the original power times the constant multiplier times the function to a power one less times the derivative of the function.

This is one of the most useful formulas in calculus, and it should be carefully remembered. In some problems it is a long process to get dy/du, so that it becomes easy to forget the factor du/dx. You should be on guard against this common error.

Example. If $y = 5\sqrt{3x^2 + 2}$, find dy/dx.
Let $u = 3x^2 + 2$. Then

$$y = 5u^{1/2}$$

$$\frac{dy}{du} = \frac{5}{2}u^{-1/2} = \frac{5}{2\sqrt{3x^2 + 2}}$$

Next, we calculate du/dx:

$$\frac{du}{dx} = \frac{d}{dx}(3x^2 + 2) = 6x$$

Applying (24),

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 6x\frac{5}{2\sqrt{3x^2 + 2}} = \frac{15x}{\sqrt{3x^2 + 2}}$$

In practice we dispense with much of this writing. *Mentally* letting $u = 3x^2 + 2$, we write only this much:

$$\frac{dy}{dx} = \frac{5}{2}\frac{1}{\sqrt{3x^2 + 2}}6x = \frac{15x}{\sqrt{3x^2 + 2}}$$

PROBLEMS

In Probs. 1 to 10 find dy/dx.

1. $y = 6(5 - x^2)^2$
2. $y = 11(x^2 + 21)^2$
3. $y = 5\sqrt{100 - x^4}$
4. $y = 2\sqrt[3]{x^2 + 9}$
5. $y = 16\sqrt[5]{x - x^2}$
6. $y = 7(x^2 + 4x + 4)^2$
7. $y = 4(x^2 + 16)^{2/3}$
8. $y = 1/\sqrt[3]{x^2 + 3}$
9. $y = 8(\sqrt{x} + 3x^2)^2$
10. $y = 3(\sqrt[3]{x} - x^{4/3})^2$

11. The charge supplied to a capacitor varied over a certain interval as $q = (t^2 - 0.0001)^3$ coulombs. Find the charging current when $t = 0.2$ second.

12. The capacitive reactance which must be connected in series with a resistance r ohms in order to produce a given total impedance Z is $X_C = \sqrt{Z^2 - r^2}$. How fast does X_C change with respect to r?

13. The effective thermal noise voltage produced across a resistor of r ohms, at an absolute temperature T, is $v = 2\sqrt{kTBr}$, where k is a constant (the Maxwell-Boltzmann constant) and B is the width in cycles of the band over which the voltage measurement is made. For any given B and T find the rate at which v varies with r.

14. In a magnetic field of intensity $\mathbf{H}$, a magnet vibrates with a period T seconds, given by $T = 2\pi\sqrt{I/M\mathbf{H}}$. ($I$ is the moment of inertia of the magnet about its center, and M is its magnetic moment.) Find $dT/d\mathbf{H}$.

15. The speed of sound in a gas is $\mathrm{v} = \sqrt{\gamma P/D}$, where γ is a constant depending upon the gas, P is the pressure of the gas, and D is its density. How fast does v vary with D?

16. According to the theory of relativity, the apparent mass m_a of a charged particle moving at speed v is

$$m_a = \frac{m_0}{\sqrt{1 - (\mathrm{v}/c)^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light in a vacuum. Get an expression for $dm_a/d\mathrm{v}$.

5-10 Derivative of a product. Let us next get a formula for the derivative of

$$y = uv \tag{25}$$

where u and v are functions of x.

To accomplish this, let x go through a change Δx, so that u and v make corresponding changes Δu and Δv. The resulting value of y is then

$$y + \Delta y = (u + \Delta u)(v + \Delta v) = uv + u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v \tag{26}$$

Subtracting (25) from (26),

$$\Delta y = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

(This could have been written in various ways. Of course, we have used a form which leads to a useful result.) If we let Δx approach zero, the left member approaches dy/dx. The first and second terms of the right member have the limits $u\,dv/dx$ and $v\,du/dx$, respectively. The third term has the limit $0\,dv/dx = 0$. Then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Substituting (25),

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{27}$$

That is,

The derivative of the product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first.

Example 1. Differentiate $y = x^3\sqrt{2x+1}$.

If we mentally consider the first factor x^3 to be the function u and the factor $\sqrt{2x+1}$ to be the function v, then Formula (27) gives

$$\frac{dy}{dx} = x^3\frac{1}{2}(2x+1)^{-\frac{1}{2}}(2) + (2x+1)^{\frac{1}{2}}(3x^2)$$

This result can be simplified somewhat if we factor out the lowest powers of x and of $2x + 1$, which are common to both terms; that is, if we factor out $x^2(2x+1)^{-\frac{1}{2}}$:

$$\frac{dy}{dx} = x^2(2x+1)^{-\frac{1}{2}}[x + 3(2x+1)] = \frac{7x^3 + 3x^2}{\sqrt{2x+1}}$$

Example 2. Differentiate $y = 2x^2(x^2+3)^3$.

In this problem, we could first multiply out the factors, then differentiate them term by term. But this often becomes tedious, even where it is possible. Differentiating according to (27), instead, we let $u = 2x^2$ and $v = (x^2+3)^3$. These substitutions, preferably made mentally, yield

$$\frac{dy}{dx} = (2x^2)(3)(x^2+3)^2(2x) + (x^2+3)^3(4x)$$

This completes the calculus part of the operation. But we can often put the solution in a neater form through some algebraic operations. In fact, this is often a major part of the work:

$$\begin{aligned}\frac{dy}{dx} &= 12x^3(x^2+3)^2 + 4x(x^2+3)^3 = 4x(x^2+3)^2[3x^2 + (x^2+3)] \\ &= 4x(x^2+3)^2(4x^2+3)\end{aligned}$$

PROBLEMS

Solve the following problems, using the formula for the derivative of a product. In Probs. 1 to 15 find dy/dx.

1. $y = x(x^2 + 1)$
2. $y = x^2(x + 1)^2$
3. $y = 2x^3(2x + 1)^2$
4. $y = (x + 1)^2(x^2 + 1)$
5. $y = (x^2 + 1)^2(x^2 - 1)^2$
6. $y = 3(x^2 - 5)(2x + 3)^3$
7. $y = (x - 2)(x^3 + 5)^4$
8. $y = (2x^3 + 5)^2(4x^2 - 3)^3$
9. $y = (9x^3 + 1)^3(4 - 3x^2)^4$
10. $y = x^2 \sqrt{x + 2}$
11. $y = 2x^2 \sqrt{x^2 + 1}$
12. $y = 5x^2 \sqrt[3]{3 - 2x^3}$
13. $y = x(x^2 + 1)(2x + 3)^2$
14. $y = (x^2 + 2)(3x + 5)^3(1 - x^3)$
15. $y = (2x^2 + 3)^2(x + 5)^2 \sqrt{1 - 2x^2}$

16. The resistance r of a resistor is varied so that $r = 10t^2 + 2$ ohms, while the current through the resistor is $i = 3t$ amperes. Find the rate of change of the voltage across the resistor.

17. The current in a resistor is $i = 2t^2 + 1$ amperes. If the resistance is $3t^3$ ohms, find the rate of change dp/dt of the power in the resistor.

18. If a conductor of length l meters moves at a speed v meters per second in a magnetic field of flux density **B** teslas, a voltage $v = -\mathbf{B}l\mathrm{v}$ volts is induced. If $l = 0.5$ meter, find a formula for dv/dt when $\mathbf{B} = 1/t^3$ and $\mathrm{v} = 100(t + 2)$.

19. The heat W in calories liberated from a resistor varied according to $W = 0.239ri^2t$. If $r = t^2/2$ ohms, and if $i = 6t + 2$ amperes, find dW/dt.

20. The thermal noise voltage across a resistor is $v = 2\sqrt{kTBr}$, where $k = 1.38 \times 10^{-23}$, T is the absolute temperature in degrees, B is the *noise bandwidth* in cycles over which the measurement is made, and r is the value of the resistance in ohms. If $B = 4 \times 10^6$, and if $T = t + 290$ and $r = 2t^2 + 1{,}000$, where t is time in seconds, how fast is v changing when $t = 10$?

21. A capacitor varied so that $C = 10^{-9}t^{-2}$ farads, while the voltage applied to it was varied according to $v = 50(t^2 - 1)$ volts. Find the rate of change of the energy $w = Cv^2/2$ stored in the capacitor when $t = 1.2$ seconds.

5-11 Derivative of a quotient. Let

$$y = \frac{u}{v} \tag{28}$$

where u and v are functions of x. To differentiate this function let x take on a change Δx, so that u and v go through corresponding changes Δu and Δv. The resulting value of y becomes

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$$

Subtracting (28),

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

Placing the right member over a common denominator,

$$\Delta y = \frac{v\,\Delta u - u\,\Delta v}{v^2 + v\,\Delta v}$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{v\,\Delta u/\Delta x - u\,\Delta v/\Delta x}{v^2 + v\,\Delta v} \tag{29}$$

We now let Δx approach zero. This results in

$$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$$
$$\frac{\Delta u}{\Delta x} \to \frac{du}{dx}$$
$$\frac{\Delta v}{\Delta x} \to \frac{dv}{dx}$$

Also, since Δv must approach zero along with Δx,

$$v\,\Delta v \to v \times 0 = 0$$

The equation relating the limits of the quantities in (29) is then

$$\frac{dy}{dx} = \frac{v\,du/dx - u\,dv/dx}{v^2}$$

Substituting (28),

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v^2} \tag{30}$$

In words,

The derivative of a quotient is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator—all divided by the denominator squared.

Example 1. Differentiate $y = x/(x^2 + 1)$.

It is true that this may be written $x(x^2 + 1)^{-1}$ and differentiated as a product. (In many problems, it is easier to do this, but here we use the *quotient* formula to obtain practice in its use.) Mentally letting $u = x$ and $v = x^2 + 1$, we apply (30):

$$\frac{dy}{dx} = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Example 2. Differentiate $y = \sqrt{2x - 3}/(x^2 + 1)^3$.

Formula (30) gives

$$\frac{dy}{dx} = \frac{(x^2 + 1)^3(\frac{1}{2})(2x - 3)^{-\frac{1}{2}}(2) - (2x - 3)^{\frac{1}{2}}(3)(x^2 + 1)^2(2x)}{(x^2 + 1)^6}$$
$$= \frac{(x^2 + 1)^3(2x - 3)^{-\frac{1}{2}} - 6x(x^2 + 1)^2(2x - 3)^{\frac{1}{2}}}{(x^2 + 1)^6}$$

Simplifying such a result is often as difficult as the original differentiation. Here

we may factor out $(x^2 + 1)^2(2x - 3)^{-1/2}$:

$$\frac{dy}{dx} = \frac{(x^2 + 1)^2}{\sqrt{2x - 3}} \frac{(x^2 + 1) - 6x(2x - 3)}{(x^2 + 1)^6}$$

In collecting terms a complicated quantity following the minus sign in the numerator is often found. Special care must be given to getting the succeeding signs correct. The final result is

$$\frac{dy}{dx} = -\frac{11x^2 - 18x - 1}{(x^2 + 1)^4 \sqrt{2x + 3}}$$

PROBLEMS

Solve the following problems, using the formula for the derivative of a quotient. In Probs. 1 to 15 find dy/dx.

1. $y = \dfrac{x}{x^2 + 1}$
2. $y = \dfrac{x^2}{1 - x}$
3. $y = \dfrac{2x^2 + 1}{x - 1}$
4. $y = \dfrac{2x + 3}{1 - x^2}$
5. $y = \dfrac{1 - 2x^2}{(2x + 5)^2}$
6. $y = \dfrac{(5x + 2)^2}{2 - 3x}$
7. $y = \dfrac{(1 - x)^2}{(2 + x^2)^2}$
8. $y = \dfrac{(x + 1)^2}{(3x + 4)^3}$
9. $y = \dfrac{(1 + x)^3}{(1 - x^2)^2}$
10. $y = \dfrac{(2x^2 + 5)^2}{(x^2 - 4)^2}$
11. $y = \dfrac{\sqrt{x^2 + 2}}{x^2 - 2}$
12. $y = \dfrac{\sqrt{x^3 - 1}}{(x^2 + 2)^2}$
13. $y = \dfrac{(3x - 5)^2}{\sqrt{2 + x^2}}$
14. $y = \dfrac{x^2 \sqrt{5x^2 + 3}}{(2 - x^2)^3}$
15. $y = \dfrac{5x^3 \sqrt{(2x^2 + 1)^3}}{3 \sqrt[5]{(x^2 - 2)^2}}$

16. The voltage applied to a resistor is $v = 2t$. If the resistance varies according to $r = t^2 - 1$, find di/dt.

17. Find the rate of change of power in the resistor of Prob. 16.

18. The distance traveled by a charged particle varied with time according to $s = (at^2 + bt + c)/\sqrt{t}$. Find a formula for its speed.

19. The work done by an electromagnet varied over a certain interval of time t according to $w = k(t^2 + 3t)/(t - 1)^2$. Find the power delivered at any instant.

20. The force $\mathbf{F}$ newtons between two charged particles separated by s meters is $\mathbf{F} = Qq/4\pi\epsilon s^2$, where Q and q are the charges in coulombs and ϵ is the dielectric constant of the medium. If Q is a constant and q varies with time according to $q = (t^2 + 1)10^{-3}$, and if $s = 0.1t$, find $d\mathbf{F}/dt$.

5-12 Further notations for the derivative. Thus far we have used the symbol dy/dx to indicate the derivative of y with respect to x. Other notations are sometimes used, and we now present certain of them.

a. The y' notation. The symbol y' (read "y prime") indicates the *derivative* of a function y with respect to x. It may be taken as being synonymous with dy/dx.

Example 1. If $y = 7x^3 + 10x$, find y'.
Differentiating, we get $y' = 21x^2 + 10$.

b. The functional notation. It has been noted that the statement that "y is equal to the f function of x" may be written $y = f(x)$. We now introduce the symbol $f'(x)$, which indicates *the derivative of the f function of x with respect to x.* That is,

$$\text{if} \quad y = f(x) \qquad \text{then} \qquad \frac{dy}{dx} = f'(x) \tag{31}$$

Example 2. If $f(x) = 2x^3 - x^5$, find $f'(x)$.
Differentiating by rule,

$$f'(x) = 6x^2 - 5x^4$$

It will be remembered that the value of $f(x)$ when x has some specific value, such as a, is indicated by $f(a)$. Similarly with the derivative notation: if $y = f(x)$, then the symbol $f'(a)$ means the value taken by the derivative $f'(x)$ when $x = a$.

Example 3. If $f(x) = 4x^2 + x^3$, find $f'(2)$.
First we obtain the derivative formula:

$$f'(x) = 8x + 3x^2$$

Then we substitute $x = 2$ in the derivative formula:

$$f'(2) = 8(2) + 3(2)^2 = 28$$

Note carefully that the differentiation is carried out *first,* then the specific value of x is substituted in the resulting derivative formula.

PROBLEMS

In Probs. 1 to 10 find y'.

1. $y = 5x$
2. $y = 2x + 8$
3. $y = 16x^2$
4. $y = 3x^2 + 5x$
5. $y = 5x^3 + x$
6. $y = 2x^2 + 6x$
7. $y = 7x^2 + 6x + 11$
8. $y = 9x^2 + 12\sqrt{x}$
9. $y = (x + 10)(x^2 - 5)^2$
10. $y = (x^2 + 2)/\sqrt{x^2 + 1}$

In Probs. 11 to 15 find $f'(x)$.

11. $f(x) = 2x^2 + 2x + 3$
12. $f(x) = 3x^3 + 60$
13. $f(x) = x + 15x^2 + 3x^3$
14. $f(x) = 4(x^3 - x)^2(x + 2)$
15. $f(x) = x^3/\sqrt{x^2 + 7}$

In Probs. 16 to 23 find the quantities required.

16. If $f(x) = x^2 - 2x + 4$, find $f'(4)$.
17. If $f(x) = x^3 - 5x^2 + 6x + 6$, find $f'(-1)$.
18. If $f(x) = x^3 + 3x^2 + x - 4$, find $f'(-4)$.

19. If $f(x) = 2x^3 - 4x^2 - 3x + 21$, find $f'(2)$.
20. If $f(x) = x^4 - 4x^3 + 6x + 2$, find $f'(2)$.
21. If $f(x) = 2x^4 + 3x^3 - x^2 + 5x$, find $f'(-1)$.
22. If $f(x) = (2x^2 + 1)(x - 1)^2$, find $f'(1)$.
23. If $f(x) = (x^2 + 2)/(3x^2 - 6)$, find $f'(-1)$.

5-13 Implicit differentiation. In Sec. 2-8 we mentioned *implicit functions*, that is, functions whose relationships with the independent variable were not stated directly in the form $y = f(x)$. As an example, consider the curve of Fig. 5-6, which is a section of a parabolic reflector for a radar antenna. Its equation is

$$y^2 = 16x \tag{32}$$

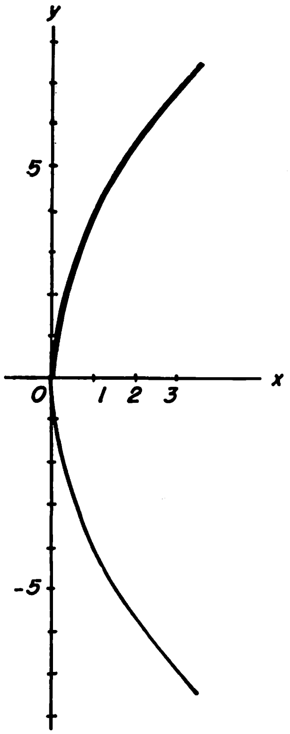


Fig. 5-6

Let it be desired to find the slope at any point of the upper "branch" of this curve (the portion drawn with the heavy line).

Notice that y is not given directly as an explicit function of x. In this particular problem we could, of course, first solve for y, getting $y = \pm 4\sqrt{x}$. We could then select the positive root in this expression, getting

$$y = 4\sqrt{x} \tag{33}$$

which represents the upper branch of the curve. We could then differentiate (33), getting the desired slope.

Another method, called *implicit differentiation*, is to differentiate (32) at once. By (24), the derivative of the left member is $2y\, dy/dx$. Thus

$$2y\frac{dy}{dx} = 16$$

$$\frac{dy}{dx} = \frac{16}{2y} = \frac{8}{y}$$

This is the desired derivative. In this result we might substitute (33), if desired, getting dy/dx as a function of x:

$$\frac{dy}{dx} = \frac{2}{\sqrt{x}}$$

The method of implicit differentiation just illustrated is commonly used when it is inconvenient or impossible to express a function explicitly in the form $y = f(x)$.

Example 1. Differentiate implicitly, obtaining dy/dx,

$$x^2 + 4xy + 4y^2 = 0$$

Taking this equation term by term, we obtain the following derivatives:

$$2x + 4\left(x\frac{dy}{dx} + y\right) + 8y\frac{dy}{dx} = 0$$

Solving for dy/dx,

$$\frac{dy}{dx} = -\frac{x + 2y}{2x + 4y} = -\frac{1}{2}$$

In problems of this kind, it is often convenient to use the y' notation, as in the following example.

Example 2. Differentiate implicitly $2y + 3x^2y^2 = 0$.
This gives

$$2y' + (3x^2)(2yy') + (y^2)(6x) = 0$$

Solving for y' yields

$$y' = \frac{-3xy^2}{1 + 3x^2y}$$

QUESTIONS

1. State the rule for obtaining the derivative of the sum of several functions.
2. State the formula for the derivative of a function of a function.
3. What rule gives the derivative of a power of a function multiplied by a constant?
4. State the rule for obtaining the derivative of the product of two functions.
5. State the rule for obtaining the derivative of the quotient of two functions.
6. If y is a function of x, what is the meaning of y'?
7. What is indicated by the function $f'(x)$?
8. What is meant by $f'(a)$ if $y = f(x)$?
9. What name is applied to the procedure for obtaining dy/dx without first obtaining y as an explicit function of x?

PROBLEMS

Solve the following problems, using implicit differentiation. In Probs. 1 to 10 find dy/dx.

1. $2y + y^2 + x^2y = 0$
2. $x^2y^2 + 10x = 0$
3. $x^2y^2 + 10y = 0$
4. $2xy^3 + y^2 = 23$
5. $x^2 + xy + y^2 = 0$
6. $x^3y + 3x^2 + 4y^2 = 0$
7. $x^2y^3 - xy + 2y = 0$
8. $5x^2 - 3y^2 + 2x^3 + x^3y^3 = 0$
9. $x^2y^3 - x^3y^2 + x^3y^3 = 0$
10. $ax^my^n + bx^py^q = 0$

11. The impedance Z of a series RL circuit is related to the resistance R and inductive reactance X_L by $Z^2 = R^2 + X_L{}^2$. If X_L is constant, find dZ/dR.

12. The plate current i_b of a certain tube varied thus with voltage v_c on the grid: $(v_c + 15)^2 = 1{,}400i_b$, where i_b was in amperes. Find g_m when $v_c = -8$ volts.

13. Einstein's photoelectric equation states that $\frac{1}{2}m\text{v}^2 = hf - p$, where m is the mass of an electron, v is the speed with which it is emitted, h is a constant (Planck's constant), f is the frequency of the incident light, and p is the *work function* of the emitting metal. Find $d\text{v}/df$.

14. A curve used in loran navigation is the *hyperbola*. Find the slope at any value of x of the hyperbola $(x/a)^2 - (y/b)^2 = 1$, where a and b are constants.

15. A series RL circuit is connected in parallel with a capacitance C. The angular frequency ω at which this circuit resonates is given by $\omega^2 = 1/LC - R^2/L^2$. Find $d\omega/dL$.

16. An amplifier tube operates into a partially reactive load, so that the load line becomes an ellipse, given by $Av_b^2 + Bv_bi_b + Ci_b^2 + Dv_b + Ei_b + F = 0$, where i_b is the plate current in amperes, v_b is the plate voltage, and A, B, etc., are constants. Solve for di_b/dv_b.

17. In laying some cable, the cable is passed over a sheave whose circumference has the equation $x^2 + y^2 = 100$. The cable leaves tangent to the sheave at a point which is located at a horizontal distance 6 units from the center. What angle does the cable make with the horizontal at this point?

5-14 Applications of Kirchhoff's laws.

a. The current law. Kirchhoff's first law, or current law, can be stated in more than one form, but we choose to say that

> The sum of the currents flowing toward any point in a circuit is, at any instant, equal to zero.

Stated as an equation,

$$\Sigma i = 0 \tag{34}$$

The symbol Σ (Greek capital letter sigma) is the symbol of *summation*, and it indicates that one is to take the sum of all the quantities of the kind indicated by the symbol which follows Σ. We could read (34) aloud by saying that "the summation of the currents flowing toward any point in a circuit is zero."

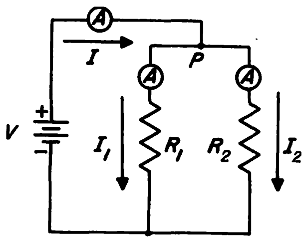


Fig. 5-7

To illustrate this law notice Fig. 5-7, where a battery supplies a fixed voltage V to a parallel combination of two resistors R_1 and R_2. Consider a point P in this figure. According to Kirchhoff's current law, the sum of the currents flowing toward this point is zero.*

* Throughout this book we use the *conventional* direction of current flow, which is from positive to negative in a circuit external to a battery, except in cases where it is necessary to consider the electron nature of the current. In the latter case we shall discuss *electron flow*, rather than current. This is the established practice of engineers and scientists and is followed in all but a few texts. (See *current* defined in "American Standard Definitions of Electrical Terms," approved by the American Standards Association and published by American Institute of Electrical Engineers, New York.)

Note that the battery current I flows *toward* point P but the resistor currents I_1 and I_2 flow *away* from this point, so we consider I_1 and I_2 as *negative* currents flowing toward P. This gives

$$\Sigma i = I - I_1 - I_2 = 0$$

For instance, if $I_1 = 2$ amperes, and if $I_2 = 3$ amperes, we should expect the battery current I to be 5 amperes, since

$$5 - 2 - 3 = 0$$

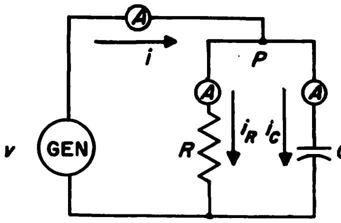


Fig. 5-8

Now consider a generator GEN (Fig. 5-8) supplying a varying voltage v of any obtainable waveform. Let this voltage be applied to a parallel resistor-capacitor combination, as in Fig. 5-8. Let i represent the generator current, flowing toward point P. The current in the resistor, by Ohm's law, is $i_R = v/R$ at all times. By (10), the capacitor current is $i_C = C\,dv/dt$. Then, by (34),

$$\sum i = i - \frac{1}{R}v - C\frac{dv}{dt} = 0$$

or

$$C\frac{dv}{dt} + \frac{1}{R}v - i = 0 \qquad \textbf{(35)}$$

Example 1. In Fig. 5-8 let $C = 10$ microfarads, $R = 100{,}000$ ohms, and $v = 2t^3 + 100$ volts. Find the current i supplied by the generator when $t = 10$ seconds.

We find $dv/dt = 6t^2$, which takes the value 600 volts per second when $t = 10$. Also, $v = 2{,}100$ volts when $t = 10$. By (35),

$$10^{-5}(6 \times 10^2) + \frac{2.1 \times 10^3}{10^5} - i = 0$$

$$i = 2.7 \times 10^{-2} \text{ ampere} = 27 \text{ milliamperes}$$

b. The voltage law. This is Kirchhoff's second law, and it may be stated

The sum of the voltage drops around a circuit is, at any instant, equal to zero.

That is,

$$\Sigma v = 0 \qquad \textbf{(36)}$$

To illustrate, Fig. 5-9 shows a battery of voltage V supplying a current I to two resistors R_1 and R_2 in series. Let us begin at the top of R_1, for example, and consider the voltage drops as we proceed in a clockwise direction around the circuit. We encounter first the voltage drop across R_1, which we shall call V_1. Next we find a drop, which we shall call V_2, existing across R_2. We take a *source* voltage drop as a *negative* voltage drop if its polarity is such as to aid the current in a clockwise direction. This gives

$$\Sigma v = V_1 + V_2 - V = 0$$

For instance, if $V_1 = 20$ volts, and if $V_2 = 50$ volts, we should expect the source voltage to be 70 volts.

Now consider the circuit of Fig. 5-10, where a generator GEN supplies a current i of any obtainable waveform to a circuit composed of the

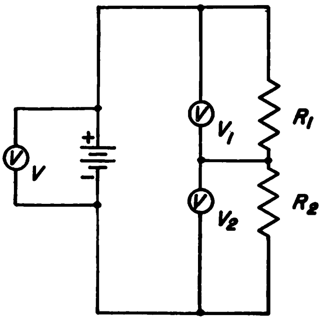


Fig. 5-9

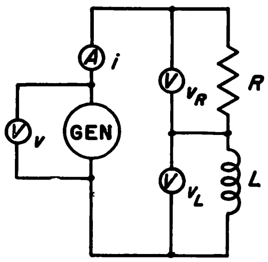


Fig. 5-10

resistance R in series with the inductance L. (To simplify our calculations imagine that R includes the resistance which must be present in the inductor.) Let v represent the generator voltage at any instant.

The voltage drop across R is, of course, $v_R = Ri$. Naturally, this voltage drop has a polarity such that it tends to *reduce* the current in the circuit. We take this as a *positive* drop.

Since we neglect the resistance of the inductor L, the voltage across L must be simply the induced emf due to the rate of change of current, $v_{ind} = -L\,di/dt$. That is, the numerical value of the voltage drop across L is always $L\,di/dt$; the minus sign is a convention which indicates that the induced emf has a polarity such as to oppose the change in current. We must now decide what algebraic sign should be attached to the voltage drop $L\,di/dt$ so that actual circuit conditions will be correctly represented. To do this, suppose for the moment that the current is increasing; that is, let di/dt be positive. Lenz's law and our experience with inductors tell us that the induced emf will tend to *reduce* the current. As in the case of the resistance R, discussed above, we consider the voltage drop

across L in such cases to be positive. Similarly, the induced emf in L will tend to *aid* the current flow if di/dt is negative, and we call such an emf a *negative* voltage drop. Thus the voltage drop across L is positive or negative according as di/dt is positive or negative. Therefore we may indicate the *voltage drop* across L simply as

$$v_L = L\frac{di}{dt} \tag{37}$$

Note that this *voltage-drop equation* expresses the voltage which we should have to impress across an inductor (of negligible resistance) in order to cause a prescribed current i to flow.*

Returning to the circuit of Fig. 5-10, we consider the various voltage drops around the circuit and equate their sum to zero in accordance with (36):

$$L\frac{di}{dt} + Ri - v = 0 \tag{38}$$

Example 2. In the circuit of Fig. 5-10 let $i = t^2 - 2t - 3$ amperes. If $L = 6$ henrys, and if $R = 8$ ohms, what must be the equation for the generator voltage v?

We find $di/dt = 2t - 2$ amperes per second. By (38),

$$6(2t - 2) + 8(t^2 - 2t - 3) - v = 0$$

or

$$v = 8t^2 - 4t - 36 \qquad \text{volts}$$

Note that the currents and voltages in the preceding examples are neither steady dc nor sinusoidal ac in form. Here, too, we are working with circuits in which *the voltages and the currents do not, in general, have the same waveforms.* You will see that the methods just brought out permit you to work some problems which would not have been solvable by the impedance formulas of elementary electricity or by any ordinary applications of vector methods. In fact, the impedance relationships appear in later studies as special cases of the present methods. Applications to more complicated circuits are treated in other texts.[1]

QUESTIONS

1. Give the meaning of the symbol Σ.
2. State Kirchhoff's current law, both as a sentence and as an equation.
3. State Kirchhoff's voltage law, both as a sentence and as an equation.
4. What equation relates the currents and voltages in a parallel RC circuit?
5. Give the equation which relates the currents and voltages in a series RL circuit.

* To summarize, the *induced emf* generated in an inductor is given by the law of Henry and Faraday and by Lenz's law as $v_{ind} = -L\,di/dt$. When considering this emf as a *voltage drop,* we get correct results by writing it as $v_L = L\,di/dt$.

PROBLEMS

1. A voltage $v = t^3 + 2{,}000$ volts is impressed across a parallel RC combination, where $R = 300{,}000$ ohms and $C = 20$ microfarads. Find the resultant current at any time t.

2. Similar to Prob. 1, except let $R = 1$ megohm and $C = 2$ microfarads.

3. A 4-microfarad filter capacitor is shunted by a bleeder resistor of 50,000 ohms. If, during the charging process, the voltage across the capacitor varies approximately as $v = 1{,}000t^{2/3} + 100$ volts, find the current supplied to the combination when $t = 0.001$ second.

4. An amplifier tube operates into a plate-load resistance of 200,000 ohms. The shunt capacitance in the circuit is 120 micromicrofarads. If, over a certain interval, the output voltage supplied by the tube varies according to $v = 2 \times 10^7 t + 200$ volts, find the plate current of the tube when $t = 10^{-5}$ second.

5. If a current $i = 3t^{1/3} + 2$ amperes is transmitted through a series RL circuit, where $R = 100$ ohms and $L = 20$ henrys, find the voltage across this circuit when $t = \frac{1}{8}$ second.

6. Similar to Prob. 5, except let $i = 10t^{1/2} + 0.1$ amperes, $R = 800$ ohms, $L = 320$ henrys, and $t = 0.04$ second.

7. A relay winding has an inductance of 0.6 henry and a resistance of 480 ohms. If the current through the coil is $i = t^{1/2} + 0.02$ ampere, what is the voltage across it when $t = 0.01$ second?

8. An amplifier tube has a plate-load resistor of 2,000 ohms, which has a compensating inductor $L = 20$ millihenrys in series with it. If the current through the combination is $i = 2.5 \times 10^4 t + 0.01$ amperes, where t is in seconds, find the voltage across the circuit when $t = 0.025$ microsecond.

9. A 30-henry inductor is connected in series with a 75-ohm resistor. If a current $i = 2t^2 + t$ is supplied through this combination, after what time t, in seconds, is the voltage across the combination equal to 375 volts?

10. A 30-microfarad capacitor is shunted by a 25,000-ohm resistor. If the applied voltage is $v = 300t^2$, at what time t, in seconds, is the total current i equal to 84 milliamperes?

11. A 40-microfarad capacitor and a 200-henry inductor are connected in series. The combination is connected to a generator. The voltage across the capacitor has the form $v_C = 1{,}000(t^3 + t + 1)$ volts. What is the voltage across the inductor when $t = 1$ second?

5-15 Differentiating circuits. *a. The RC differentiator.* Figure 5-11 shows a circuit containing a small capacitance C in series with a small resistance R, driven by a constant-voltage generator such as a tube having only a small value of plate resistance. Neglecting the effect of R, the current in this circuit is, by (10),

$$i_C = C\frac{dv}{dt}$$

The voltage drop across R, which is here used as the output signal, is

$$v_R = Ri_C = RC\frac{dv}{dt} \tag{39}$$

Accordingly, the output voltage of this circuit is proportional to the derivative of the input voltage, so that the circuit is called a *differentiating circuit.* Such circuits have wide application. If, for example, we supplied this circuit with a rectangular input wave, such as a television horizontal synchronizing pulse (Fig. 5-12*a*), the output waveform would be that of Fig. 5-12*b*. At the leading edge of the input pulse (time A),

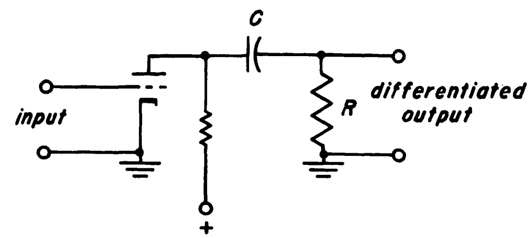


Fig. 5-11

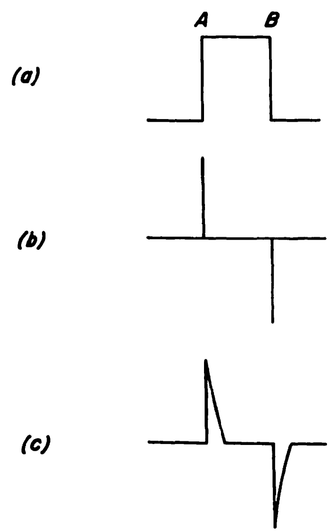


Fig. 5-12

the rate of change of voltage is briefly very great, producing a large output voltage *spike.* Along the flat top of the input pulse the derivative dv/dt is zero, producing zero output voltage. When the input wave drops very rapidly, at time B, a negative spike of output voltage is produced, signifying a large rate of decrease in the input voltage. (In practice, the differentiation is not perfect, since resistance is present. Thus, a certain amount of time is required to charge and discharge the capacitor. The differentiated wave will more closely resemble Fig. 5-12*c*.)

b. The inductive differentiator. Figure 5-13 shows a second form of differentiating circuit. Here an inductance L is fed by a constant-current generator, such as a pentode or other tube having a large plate resistance. The resistance R_L of the coil is shown separately for the purpose of considering the operation of the circuit with a theoretically perfect inductor. By (37), the voltage drop across the inductor is

$$v_L = L\frac{di}{dt}$$

and we use this voltage drop as the output voltage of the circuit. Thus, the output voltage is proportional to the derivative of the input signal.

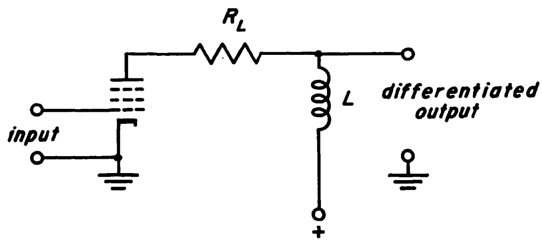
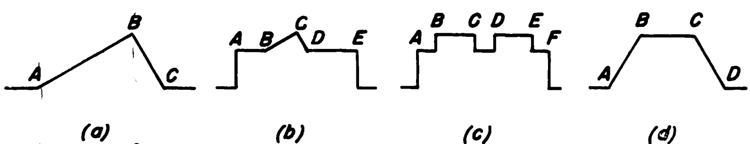


Fig. 5-13

The operation can be illustrated in a manner similar to that of Fig. 5-12. Here again, the differentiation is imperfect in practical circuits, partly because we cannot separate the resistance of the inductor from its inductance.

PROBLEM

1. Sketch or trace the waveforms of Fig. 5-14, and under each show the waveform you would expect to get after differentiation. Assume perfect differentiator action.



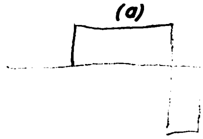


Fig. 5-14

REFERENCE

1. H. PENDER and S. R. WARREN: "Electric Circuits and Fields," McGraw-Hill Book Company, Inc., New York, 1943.

6

Differentials

Thus far we have looked upon dy/dx as a single quantity. It is useful, however, to consider dy and dx separately, as is done in this chapter.

6-1 Differentials. Let y be a function $f(x)$, and let $dy/dx = f'(x)$ be the derivative of y with respect to x. Now

➤ Let dy and dx represent two quantities, large or small, such that their ratio is equal to this derivative. Then dy is called the differential of y and dx is called the differential of x.

That is,

➤ $$\text{Differential } dy \div \text{differential } dx = \text{derivative } \frac{dy}{dx} = f'(x) \qquad \textbf{(1)}$$

It is important to observe that we do not, at least in a strict sense attempt to assign any numerical values to differentials. The quantities dy and dx in the derivative dy/dx are simply *any* two quantities whose ratio is the derivative, and, as stated, these quantities may be either large or small.

Applying the above definition to the relation

$$\frac{dy}{dx} = f'(x) \qquad (2)$$

we may multiply both members by dx, getting

$$dy = f'(x)\,dx \tag{3}$$

This result allows us to express the differential dy of a function y in terms of the independent variable x and its differential dx. The quantity $f'(x)$ in Formula (3) is called the *differential coefficient.*

Example. If $y = 4x^3$, find dy.
We get

$$\frac{dy}{dx} = 12x^2$$

Multiplying by dx,

$$dy = 12x^2\,dx$$

(NOTE: A quantity having a differential as one of its factors is also a differential. In this result, for instance, $12x^2\,dx$ is a differential, namely, the differential of y.)

To obtain an equation for the differential of a quantity we may first calculate its derivative, then solve for the required differential.

The process of finding either a derivative or a differential is called *differentiation.*

QUESTIONS

1. What is meant by the terms *differential of y* and *differential of x* in the derivative expression dy/dx?
2. Is the expression $2x^2\,dx$ indicative of a differential?
3. Is the quantity $(x^2 - 5x - 4)\,dx$ a differential?
4. What term is applied to the process of finding a derivative? A differential?

PROBLEMS

In Probs. 1 to 15 find dy.

1. $dy/dx = 11x^5$
2. $dy/dx = 2x^2 + 1$
3. $dy/dx = x + 100x^2$
4. $dy/dx = x^2 + 2x + 5$
5. $dy/dx = 2x^2 + 12x - 13$
6. $y = ax^3 - b$
7. $y = 5x^2 + 11x - 7$
8. $y = x^3 + 3x$
9. $y = 2x^3 - 11x^2 + 4$
10. $y = 9x - 2x^2 + x^3$
11. $y = 2 - x + 7x^2$
12. $y = 3 - x + 2x^2 - x^3$
13. $y = x - 1{,}000x^3$
14. $y = (x - 1)^2(2x + 5)$
15. $y = ax^m + bx^n + cx^p$

16. If $v = ir$, where r is constant, find dv.
17. If $p = i^2r$, where r is constant, find dp.
18. If $q = Cv$, where C is a constant, find dq.
19. If $w = i^2L/2$, where L is constant, find dw.

In Probs. 20 to 25 simplify the given expressions.

20. $\dfrac{dy}{dt}\dfrac{dt}{dx}$ **22.** $\dfrac{dv}{di}\dfrac{di}{dt}$ **24.** $\dfrac{dq}{dv}\dfrac{dv}{dt}$

21. $\dfrac{dw}{dv}\dfrac{dv}{dt}$ **23.** $\dfrac{dy}{dv}\dfrac{dv}{dx}$ **25.** $\dfrac{dv}{dw}\dfrac{dw}{dx}$

By what would you multiply

26. $\dfrac{dV}{dx}$ to get $\dfrac{dV}{dt}$? **28.** $\dfrac{dr}{d\theta}$ to get $\dfrac{dr}{d\phi}$? **30.** $\dfrac{dp}{di}$ to get $\dfrac{dp}{dt}$?

27. $\dfrac{dy}{dt}$ to get $\dfrac{dy}{dz}$? **29.** $\dfrac{dZ}{dx}$ to get $\dfrac{dZ}{dL}$?

In Probs. 31 to 35 express the given relations in derivative form.

31. $dy = 21x^2\,dx$
32. $dy = 3(x^2 - 1)\,dx$
33. $dq = 2x(x - 1)^3\,dx$
34. $dw = (2t - t^3)\,dt$
35. $d\theta = (\phi^2 + 5\phi - 11)\,d\phi$

36. Verify, by operations with differentials, the formula

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

37. It may be shown that under appropriate conditions

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

Verify this result by operations involving differentials.

38. If x and y are functions of t, then under suitable conditions

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Using differentials, verify this formula.

39. If $y = uv$, where u and v are functions of x, find dy. [HINT: Begin with Formula (27), Sec. 5-10.]

40. If $y = u/v$, where u and v are functions of x, find dy. [HINT: Begin with Formula (30), Sec. 5-11.]

41. If $y = uvw$, where u, v, and w are functions of x, find dy. [HINT: Use Formula (27), Sec. 5-10.]

6-2 Applications of differentials. The *differential* idea is of great value in mathematical analysis, and is used extensively in calculus and in higher courses. In addition, the differential *notation* is used for obtaining rapidly the numerical solutions to many problems which would otherwise be difficult and tedious. This use of the differential notation will now be shown.

6-3 Approximate increments. Let y be a function $f(x)$. We may indicate its derivative by

$$\frac{dy}{dx} = f'(x) \tag{4}$$

It will be recalled that $dy/dx = \lim_{\Delta x \to 0} (\Delta y/\Delta x)$, so that

➤ In case the interval Δx is very small, $\Delta y/\Delta x$ will not be much different from dy/dx [or $f'(x)$].

We write

$$\frac{\Delta y}{\Delta x} \approx f'(x) \qquad \text{if } \Delta x \text{ is very small*} \tag{5}$$

This can be written

➤ $$\Delta y \approx f'(x)\,\Delta x \qquad \text{if } \Delta x \text{ is very small} \tag{6}$$

Example. If $y = x^6$, how much will y change if x changes from 3 to 3.0002? Differentiating, $dy/dx = f'(x) = 6x^5$. Applying (6),

$$\Delta y \approx 6x^5\,\Delta x$$

Letting $x = 3$ and $\Delta x = 0.0002$,

$$\Delta y \approx 6(3)^5(0.0002) = 0.2916$$

[True, this result is only approximate. But to get an exact answer, we should have had to multiply out laboriously $(3.0002)^6$, and subtract 3^6. Such an operation gives $\Delta y = 0.291638^+$—almost the same as the result we got so quickly by Formula (6).]

In the example above, we substituted for x in Formula (6) the value of x at the *beginning* of the interval Δx, namely, $x = 3$. Our result would not be much changed by substituting instead $x = 3.0002$, the value taken by x at the *end* of the interval Δx. But the calculations would have been much more laborious. To simplify computations, in any problem we substitute for x in Eq. (6) the *most convenient* value of x in the interval Δx or even in the immediate neighborhood of the interval.

6-4 Using the differential notation. In practice the relation

$$dy = f'(x)\,dx \tag{3}$$

would be used to write the solution of a problem like that in the preceding example. In such problems we customarily avoid the symbol Δ by using dy and dx to represent the increments Δy and Δx. It must be kept in mind, however, that dy and dx are actually only symbols representing quantities *large or small* which are related by Formula (1). The use

* The symbol $\approx$ is read "is approximately equal to."

which we shall now make of the symbols dy and dx is not strictly proper, but it is customary and is considered acceptable so long as we understand what is actually meant.

To illustrate, let us present the preceding example in a form which would more likely be used in practice.

Example 1. If $y = x^6$, how much will y change if x changes from 3 to 3.0002? Differentiating, $dy/dx = f'(x) = 6x^5$. Thus

$$dy = 6x^5\,dx$$

Letting $x = 3$ and $dx = 0.0002$,

$$dy = 6(3)^5(0.0002) = 0.2916$$

You should carefully compare Example 1 with the example in Sec. 6-3. You must understand that in a strict sense we do not assign numerical values to differentials. But an increment, such as Δy, may be assigned a numerical value, and as a matter of convenience we often write increments as if they were differentials. When we see a solution in the form of Example 1 above, what is actually meant is a solution as written in the example of Sec. 6-3.

Example 2. Suppose the current in a 10-ohm resistor to vary according to $i = t^8$ over a certain interval. What change in voltage v across the resistor occurs from $t = 0.999$ to $t = 1.001$ seconds?

We write

$$v = Ri = 10t^8$$

$$\frac{dv}{dt} = 80t^7$$

$$dv = 80t^7\,dt$$

Here, the most convenient value of t in the interval from $t = 0.999$ to $t = 1.001$ is $t = 1$. Substituting this value and putting $dt = 0.002$,

$$dv = 80(0.002) = 0.16 \text{ volt}$$

Example 3. An electronic device was housed in a cubical container. The cube was actually 9.97 inches along each edge, but a small error in measurement resulted in the edge being taken as 10 inches. What was the resulting approximate error in the calculated volume?

The problem is to find how much change dV would occur in the volume because of a change $ds = -0.03$ inch along the edge.

$$V = s^3$$

$$\frac{dV}{ds} = 3s^2 \qquad \text{so} \qquad dV = 3s^2\,ds$$

Therefore $$dV = 3(10)^2(-0.03) = -9 \text{ cubic inches}$$

PROBLEMS

1. If the resistance r ohms in a circuit varies with time (t seconds) according to $r = 100 + t^{1/2}$, what approximate change dr in r occurs as t changes from 4 to 4.001?

2. The current i amperes in a circuit varied with time (t seconds) in accordance with $i = t^2 + 3t$. About what current change di occurred as t changed from 0.98 to 1 second?

3. An object falls a distance s feet in a time t seconds according to $s = gt^2/2$, where $g = 32$. Approximately what distance ds does the object fall in the period from 11 to 11.02 seconds?

4. The power in a circuit is given by $p = Ri^2$ watts, where $R = 100$ ohms and i is the current in amperes. If i changes from 12 to 12.005, approximately what change dp occurs in the power in watts?

5. The voltage v required to produce a power p watts in a resistor $R = 25$ ohms is $v = (pR)^{1/2}$. About what change dv in v is required to produce a change in p from 400 to 401 watts?

6. The impedance in an ac circuit is $z = (r^2 + X^2)^{1/2}$ ohms, where r and X are, respectively, the resistance and the reactance of the circuit. If $r = 200$ ohms and $X = 150$ ohms, what approximate change dz in z results when r changes to 202 ohms?

7. The intensity J of the heat radiation from a transmitting tube plate varies with its absolute temperature according to $J = \sigma T^4$, where σ is a constant and T is the temperature. If $J = 50$ units when $T = 1200$ degrees, approximately what change dJ units in J results from a change in T to 1205 degrees?

8. A plate from a variable capacitor is a semicircle with radius R, from which a smaller concentric semicircle of radius r has been cut to allow for the shaft. If $R = 10$ centimeters, find the approximate change dA in the area A of the plate resulting from a decrease in r from 2 to 1.99 centimeters.

9. The power in a circuit was $p = 6 - t$ watts. What was the approximate energy dw in joules expended from $t = 4$ to $t = 4.002$ seconds?

10. The mutual conductance of a tube was $g_m = 8{,}000$ micromhos. Find the approximate plate-current change di_b resulting from an increase in grid voltage from $v_c = -10$ to $v_c = -9.97$ volts.

11. The current in a circuit varied as $i = 3t + 2$ amperes. What approximate charge dq in coulombs was transmitted from $t = 1$ to $t = 1.001$ seconds?

12. The induced emf in an 8-henry inductor varied according to $v_{ind} = 3t^2 - t$. About how much change di occurred in the current i in the inductor from $t = 2$ to $t = 2.01$ seconds?

13. The current in a 9-microfarad capacitor varied according to $i = t - t^2$ amperes. Find the approximate change dv in the voltage across the capacitor from $t = 0.1$ to $t = 0.101$ second.

14. The mutual inductance between the windings of a transformer was $M = 3$ henrys. The induced secondary emf was $v_2 = 50t^{1/2}$ volts. Find the approximate change di_1 in the primary current from $t = 0.09$ to $t = 0.091$ second.

15. A particle of mass m moving at a speed v has a kinetic energy $w = \frac{1}{2}m\mathrm{v}^2$. What approximate increase dw in its energy occurs if there is a small change $d\mathrm{v}$ in the speed?

16. The low-frequency inductance of a single-layer solenoid is approximately $L = kDn^2$, where k is a *form factor*, D is the diameter in inches, and n is the number of turns. Find the approximate change dL in the inductance resulting from the addition of a small part of a turn dn.

17. The *illuminance* of a surface in a televised scene is $E = I/r^2$ foot-lamberts,

when a source of I candlepower is located r feet from the surface. Get an equation for the approximate change dE in illuminance occurring when a small change dr is made in r.

18. A right circular cone used in constructing a broadband antenna has a volume $V = \pi r^2 h/3$, where r is the radius of the base and h is the altitude. If $r = 5$ inches and $h = 12$ inches, what approximate change dV in the volume occurs when r is changed to 5.026 inches?

19. In Prob. 18, what approximate change dA in the lateral area A of the cone occurs when r changes from 5 to 5.026 inches? ($A = \pi rs$, where s is the slant height of the cone.)

20. An increase in the apparent mass m_a of a moving particle occurs in accordance with $m_a = m_0/[1 - (\mathrm{v}/c)^2]^{1/2}$, where m_0 is the mass of the particle at rest, v is its speed, and c is the speed of light in a vacuum. What approximate change dm_a occurs in the apparent mass as a result of a small change $d\mathrm{v}$ in the speed of the particle?

6-5 Conclusion. The presentation which has been given is sufficient for the uses of differentials encountered in this book. For a further treatment, see any formal calculus text.

It appears that the interpretation placed upon the differential concept varies somewhat, depending upon the use to which it is put. The references which follow may prove useful to you at a later point in your mathematical career.

REFERENCES

1. W. R. Ransom: Bringing in Differentials Earlier, *Am. Math. Monthly*, **58**(5): 336–337 (May, 1951).
2. M. K. Fort, Jr.: Differentials, *Am. Math. Monthly*, **59**(6):392–395 (June-July, 1952).
3. C. G. Phipps: The Relation of Differential and Delta Increments, *Am. Math. Monthly*, **59**(6):395–398 (June-July, 1952).
4. H. J. Hamilton: Toward Understanding Differentials, *Am. Math. Monthly*, **59**(6):398–403 (June-July, 1952).
5. C. B. Allendoerfer: Editorial, *Am. Math. Monthly*, **59**(6):403–406 (June-July, 1952).

7

Higher Derivatives

In the following pages we consider the differentiation of a function two or more times in succession.

7-1 Average acceleration. Consider an object, such as a charged particle, moving along a straight line at a speed $\mathbf{v} = 1{,}000$ meters per second. Let the speed now be increased, so that after 10 seconds $\mathbf{v} = 1{,}500$ meters per second. Then, as an average, the speed has increased by 50 meters per second for each second of this interval. We say that the *average acceleration* of the object over the 10-second interval is 50 meters per second per second.

The average acceleration of an object moving in a straight line is

$$\mathbf{a}_{\text{av}} = \frac{\Delta \mathbf{v}}{\Delta t} \tag{1}$$

where $\mathbf{v}$ is its speed at any instant.

7-2 Instantaneous acceleration. Here we shall interest ourselves not so much in the average acceleration of an object as in its *instantaneous acceleration*, defined as its *rate of change of speed* (or velocity) *at any instant.* In derivative notation the acceleration of an object at any

instant is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{2}$$

Example. A particle moves along a straight line so that its speed is $\mathbf{v} = 200t - t^2$ feet per second. Find a formula for its acceleration at any instant.

Differentiating the given formula, we find the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 200 - 2t = 2(100 - t) \qquad \text{feet per second per second}$$

7-3 Second derivatives. Now suppose that we are given a formula for the distance s moved by an object, rather than for its speed $\mathbf{v}$. We can differentiate the equation for s, getting a formula for $\mathbf{v}$ as a function of time t. And the formula for $\mathbf{v}$ can then be differentiated in turn, giving us the acceleration $\mathbf{a}$. That is,

$$\mathbf{a} = \frac{d}{dt}\mathbf{v}$$

or

$$\mathbf{a} = \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

The right member is customarily represented by the symbol d^2s/dt^2, which may be read aloud "*d* two *s* over *dt* squared." Thus we write

$$\mathbf{a} = \frac{d^2s}{dt^2} \tag{3}$$

The quantity d^2s/dt^2 is called *the second derivative of s with respect to t*. Note that it is obtained by first differentiating s with respect to t, getting the first derivative ds/dt, then differentiating this result in turn, getting d^2s/dt^2. [Notice that this is quite different from the *square* of the first derivative, which would be $(ds/dt)^2$.] The second derivative of a function, or the rate of change of its derivative, is sometimes called its *flexion*. The relation (3) can be stated

The acceleration of an object moving along a straight line is equal to the second derivative of the distance which it travels with respect to time.

Example 1. A falling object moves a distance s feet in time t seconds, according to $s = 16t^2$. Find its acceleration.

Differentiating twice in succession,

$$\mathbf{v} = \frac{ds}{dt} = 32t \qquad \text{feet per second}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 32 \text{ feet per second per second}$$

Since we considered *downward* as the positive direction in Example 1,

this result shows that a falling object increases its speed by 32 feet per second (about 9.8 meters per second) for each second of fall. This figure is approximately correct for small objects. For greater accuracy, it would have to be modified to allow for such factors as air resistance, variations in the object's distance from the center of the earth, etc. For simplicity, when considering falling objects, in this book we shall usually omit effects due to these factors and take the acceleration due to gravity as simply 32 feet per second per second, or 9.8 meters per second per second (referred to by the symbol g).

The idea of the second derivative can be applied to quantities other than distances, speeds, and accelerations. In general terms,

➡ The rate of change (with respect to x) of the derivative of a function y is expressed by d^2y/dx^2 and is called the second derivative of y with respect to x.

Example 2. The charge transmitted through a circuit varied with time according to $q = t^3 + t$ coulombs. If the circuit included an inductor of 7 henrys, find the induced emf across the inductor when $t = 3$ seconds.

Differentiating, we find that the current in the circuit is

$$i = \frac{dq}{dt} = 3t^2 + 1 \qquad \text{amperes}$$

The rate of change of i is found by a second differentiation:

$$\frac{di}{dt} = \frac{d^2q}{dt^2} = 6t \qquad \text{amperes per second}$$

This gives

$$v_{ind} = -L\frac{di}{dt} = -7(6t) = -42t \qquad \text{volts}$$

Here we shall consider an expression like d^2y/dt^2 as a single quantity. We shall make no attempt to break up such a derivative into separate parts.

7-4 Newton's second law of motion. An important relation states that

➡ The force necessary to impart a given acceleration to an object is proportional to the amount of acceleration and to the mass of the object.*

➡ $$\mathbf{F} = m\mathbf{a} \tag{4}$$

* In using this equation it is essential that the quantities be measured in units which are consistent with each other. If m is in kilograms and $\mathbf{a}$ is in meters per second per second, then $\mathbf{F}$ is in newtons (1 kilogram is about 2.2046 pounds of mass, and 1 newton is about 0.2248 pound of force). If m is in slugs and $\mathbf{a}$ is in feet per second per second, then $\mathbf{F}$ is in pounds (1 slug is about 32.17 pounds of mass). For other systems of units, see any standard physics text.

This is called Newton's second law of motion. Sir Isaac Newton published it, but he did not claim it as original.

Example. A solenoid moves a plunger a distance s meters in a time t seconds according to $s = 2t^3 + 0.02t$. If the plunger has a mass of 0.05 kilogram, what force is being applied by the solenoid when $t = 0.1$ second?

We find $\mathbf{a} = d^2s/dt^2 = 12t$. When $t = 0.1$, this gives $\mathbf{a} = 1.2$ meters per second per second. According to (4), $\mathbf{F} = m\mathbf{a} = 0.05(1.2) = 0.06$ newton.

QUESTIONS

1. What is meant by the *acceleration* of an object at any instant?
2. How would you calculate the second derivative of a function v with respect to t?
3. Read aloud d^2i/dt^2, d^2w/dt^2, d^2r/dT^2, $d^2\theta/d\phi^2$.
4. State Newton's second law of motion, both in the form of a sentence and as an equation.

PROBLEMS

In Probs. 1 to 15 find d^2y/dx^2.

1. $y = x^4 - 5$

2. $y = 5x^2 + x$

3. $y = 100x^3 + 20$

4. $y = 2x - 1{,}000$

5. $y = 5x^5 - x^6$

6. $y = 14x^2 + 5x - 8$

7. $y = 1 + x - x^2 - 2x^3$

8. $y = 10x^{10} + 2x$

9. $y = 6x^3 + 5x^2 - 7x + 2$

10. $y = ax^2 + bx + c$

11. $y = ax^m - bx^n$

12. $y = (x + 2)(x - 1)^2$

13. $y = \sqrt{2x + 3}$

14. $y = \dfrac{x + 2}{\sqrt{x}}$

15. $y = \dfrac{x^2}{\sqrt{x - 9}}$

16. The charge in coulombs transmitted through a $\frac{1}{2}$-henry inductor varied as $q = 0.5t^4 - t$. Find the induced emf across the inductor when $t = 2$ seconds.

17. The energy dissipated by a resistor was $w = 3t^3 + 4t$ joules. How fast was the power changing, in watts per second, when $t = 5$ seconds?

18. A 125-turn coil is placed in a varying magnetic field. The flux through the coil is $\phi = 0.3t^3 + 0.2t$ webers. How fast is the induced emf changing when $t = 1.5$ seconds?

19. A charge $q = 10t^3 + 2t + 1$ coulombs is sent through the primary of a transformer. If the mutual inductance between the primary and the secondary windings is $M = 18$ henrys, find the secondary emf when $t = 0.05$ second.

20. An electron has a mass of 9.1×10^{-31} kilogram. If an electric field accelerates the electron at a rate of 1.2×10^{16} meters per second per second, find the force in newtons being exerted upon it.

21. Similar to Prob. 20, except for an α particle, whose mass is 6.688×10^{-27} kilogram, if it is accelerated at 3.2×10^{15} meters per second per second.

22. An electron is set free in an electric field of intensity $\mathbf{E} = 50{,}000$ volts per meter. If the resulting force on the electron is $\mathbf{E}e$ newtons, where e is the electron charge, find the acceleration imparted to the electron. (The mass of an electron is 9.1×10^{-31} kilogram, and its charge is 1.59×10^{-19} coulomb.)

23. A proximity fuze attached to a projectile is subjected, in discharging a weapon, to an acceleration of 80,000 meters per second per second. Find the resulting force exerted upon a component whose mass is 0.002 kilogram.

24. The voltage across a 20-microfarad capacitor was varied according to $v = 2{,}000t^3 + 10t$. If a 100-henry inductor was connected in series with the capacitor, the induced voltage across the inductor had what value when $t = 2$ seconds?

25. The charge transmitted through a 12-henry inductor varies as $q = t^4 + 0.5t + 2$ coulombs. How fast was the stored energy in the field of the inductor changing when $t = 0.5$ second?

7-5 Derivatives of higher order. Just as the second derivative of a function is obtained by differentiating twice in succession, derivatives of

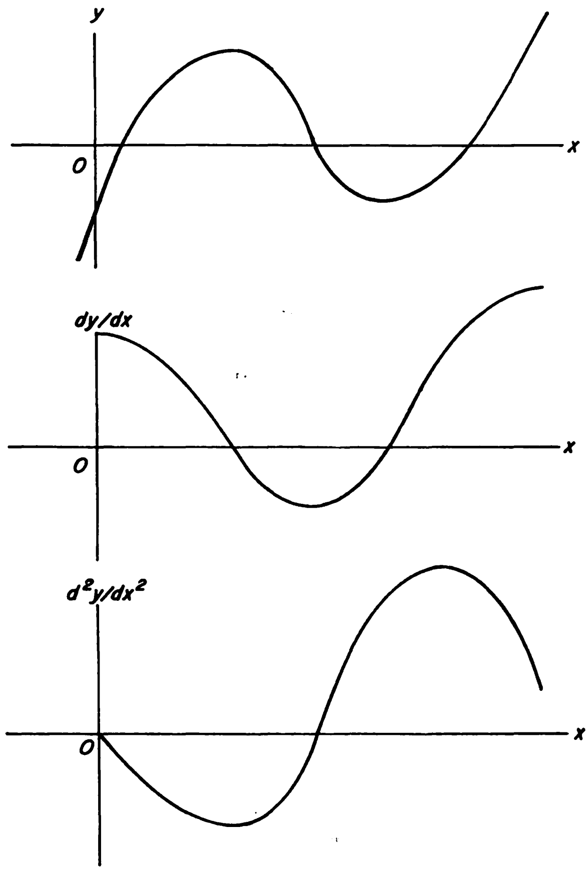


Fig. 7-1

still higher order are yielded by further differentiations. For instance, the third derivative of a function is given by differentiating the second derivative, etc. The mth derivative of a function y, with respect to x, can be indicated by d^my/dx^m.

Example. If $y = x^6 + 2x^3 + 3x$, find (a) d^3y/dx^3 and (b) d^4y/dx^4.
Successively differentiating,

$$\frac{dy}{dx} = 6x^5 + 6x^2 + 3$$

$$\frac{d^2y}{dx^2} = 30x^4 + 12x$$

$$\frac{d^3y}{dx^3} = 120x^3 + 12$$

This is the answer to part (a) of the problem. The result for part (b) is given by a further differentiation:

$$\frac{d^4y}{dx^4} = 360x^2$$

Keep in mind that each of the successive derivatives is a function of the independent variable x. This is illustrated by the graphs of Fig. 7-1. A given function y is shown in the upper curve, and its successive derivatives are graphed in the *derived curves* below. Observe that the *height* of each of these curves at any point indicates the *slope* of the curve next above it.

7-6 Other notations for the higher derivatives. As we have seen, if $y = f(x)$, we can indicate dy/dx by the symbol y'. Similarly, the second derivative d^2y/dx^2 can be shown by the symbol y'' (read "y double prime"). Likewise, the following symbols represent further derivatives:

$$y''' \text{ (read ``}y\text{ triple prime'')} = \frac{d^3y}{dx^3}$$

$$y^{\text{iv}} \text{ (read ``}y\text{ four prime'')} = \frac{d^4y}{dx^4}$$

$$y^{\text{v}} \text{ (read ``}y\text{ five prime'')} = \frac{d^5y}{dx^5} \qquad \text{etc.}$$

Example 1. Given that $x^2 + xy + y^2 = 0$, find y''.
Differentiating twice implicitly,

$$2x + xy' + y + 2yy' = 0$$
$$2 + xy'' + y' + y' + 2yy'' + 2y'^2 = 0$$

Collecting terms and solving for y'',

$$y'' = -2\,\frac{1 + y' + y'^2}{x + 2y}$$

The *functional notation* may also be used for the higher derivatives. Thus, $f''(x)$ (read "the f-double-prime function of x" or "f double prime of x") indicates the second derivative of $f(x)$. And $f'''(x)$ indicates the

third derivative of $f(x)$, and $f^{iv}(x)$ indicates the fourth derivative of $f(x)$, etc.

To indicate the value of the second derivative of $f(x)$ at the point on the graph of this function where x is equal to, say, 3, we write $f''(3)$. This is found by differentiating $f(x)$ twice, *then* substituting $x = 3$ in the result. Similarly for the values of further derivatives at particular points.

Example 2. If $f(x) = x^4 - 3x^3$, find $f'''(2)$.
We find

$$f'(x) = 4x^3 - 9x^2$$
$$f''(x) = 12x^2 - 18x$$
$$f'''(x) = 24x - 18$$

Putting $x = 2$ in the last equation, we find

$$f'''(2) = 24(2) - 18 = 30$$

PROBLEMS

1. If $y = x^2 - x^3$, find y'''.
2. If $y = x^2 + x$, find y'''.
3. If $y = x^5 - x^3$, find y^{iv}
4. If $y = (2 - x)^2$, find y'''.
5. If $y = 1/x$, find y^{iv}.
6. If $y = 1/\sqrt{x}$, find y'''.
7. If $y = x^4 + x^2 + 3x$, find y'''.
8. If $y = \sqrt{1 - x^2}$, find y''.
9. If $y = 12x^6 - 3x^4$, find y'''.
10. If $y = x^7 - 20x^5$, find y^{v}.
11. If $xy + y^2 = 5$, find y''.
12. If $xy + 6 = 0$, find y'''.
13. If $x^{2/3} + y^{2/3} = 20$, find y''.
14. If $x^2 - xy^2 - y^2 = 0$, find y''.
15. If $x^4 + y^4 - 4axy = 0$, find y'''.
16. If $f(x) = 2x^5 - 20x^3$, find f''' (1).
17. If $f(x) = x^4 - 20$, find f''' (2).
18. If $f(x) = 3x^4 + x^5$, find f''' (-1).
19. If $f(x) = 10x^4 + 10{,}000x^2$, find f''' (3)
20. If $f(x) = x^4 + x^3 - 2x^2$, find f''' (2).
21. If $f(x) = 0.2x^5 - 0.1x^3$, find f^{iv} (1).
22. If $f(x) = 2/x$, find f^{iv} (-1).
23. If $f(x) = 10x^9 - x^{-3}$, find f^{v} (1).
24. If $f(x) = (x + 1)(x^2 - 3)$, find f''' (-1).
25. If $f(x) = (x - 1)^2(x^2 + 2)$, find f''' (-2).

8

Maxima and Minima

The differentiation procedures we have learned make it possible to find the maximum and minimum values reached by many kinds of functions.

8-1 Ordinary maxima and minima. Figure 8-1 is a graph of a function $y = f(x)$. We see that this curve rises smoothly as we proceed from left to right, that is, as x increases, until we come to the point where $x = a$. For values of x greater than a the curve falls smoothly. We see that

➢ In the immediate neighborhood of $x = a$ the function y increases for values of x smaller than a, but y decreases for values of x greater than a. We say that y has a *maximum* where $x = a$.

Now we turn our attention to the point where $x = b$.

➢ In the neighborhood of this point the function y decreases for values of x smaller than b, but y increases for values of x greater than b. We say that y has a *minimum* where $x = b$.

Note that in other regions of the graph y may have values still greater than that where $x = a$ and y may have values smaller than that where $x = b$. But we do not necessarily consider these values as maxima or

minima. For our purposes, a *maximum* or a *minimum* point on the graph of a function will have meaning as defined above. The very greatest value reached by a function, in the range under consideration, can be called its *absolute maximum* value, and the very smallest value it reaches can be called its *absolute minimum* value.

Most of the functions of interest in electricity make *smooth* approaches to any maxima or minima they may have and make *gradual* changes past these maxima or minima, in the manner of the function just studied.

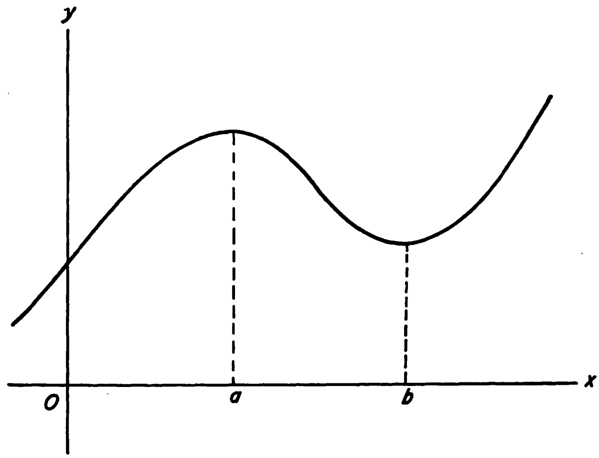


Fig. 8-1

Such maxima and minima are called *ordinary* maxima and minima to distinguish them from other kinds sometimes studied.[1–3] In this book, the terms *maximum* and *minimum* refer to *ordinary* maxima or minima, unless otherwise stated.

8-2 Horizontal tangents. In Fig. 8-2 we find three kinds of situations in which the *tangent line* to a graph may be *horizontal:*

1. At point A the function which is graphed goes through a maximum, and at this point, where the graph turns from rising to falling, its slope is zero. That is, a tangent line drawn to the graph at A is horizontal.

2. At point B, the function goes through a minimum, and at this point, also, a tangent line drawn to the graph is horizontal.

3. Points like C and D are called *points of inflection.* The graph rises on both sides of C, but its slope is decreasing just to the left of C and increasing just to the right. On the other hand, the graph falls on each side of D, but its downward or negative slope is decreasing just to the left of D and increasing just to the right. As shown, horizontal tangents may occur not only at maxima or minima but also at points of inflection.

8-3 Locating maximum and minimum points. The derivative of a function is equal to zero at a point where the graph of the function has a horizontal tangent. This gives us a method for locating possible maxima and minima of a function $y = f(x)$. We first differentiate the equation of the function; then we find the values (if any) of the independent variable x which make the derivative equal to zero. (Such values of x may be called *test points.*) We finally apply some form of *test* (to be described) to determine whether each of these test points represents a maximum, a minimum, or possibly only a point of inflection.

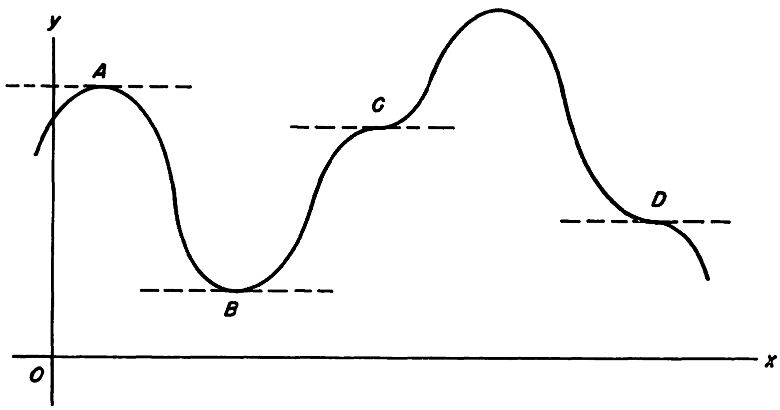


Fig. 8-2

As an example, let us find the maxima and minima, if any, of the function

$$y = 2x^2 - 8x - 4 \tag{1}$$

Differentiating,

$$\frac{dy}{dx} = 4x - 8 = 4(x - 2) \tag{2}$$

To locate test points we set this derivative equal to zero and solve for x:

$$4(x - 2) = 0$$
$$x = 2 \tag{3}$$

To find out what actually happens when $x = 2$, we can substitute values of x slightly larger and slightly smaller than 2 in the derivative formula (2). If there is a maximum at $x = 2$, then in the neighborhood of this point the derivative must be positive for values of x smaller than 2 and negative for x greater than 2, indicating that a graph of y rises to a maximum, then falls. But if dy/dx is negative for values of x smaller than 2 and positive for x greater than 2, then there must be a minimum

point where $x = 2$. Applying this test,

$$\text{When } x = 1.9 \qquad \frac{dy}{dx} = 4(1.9) - 8 = -0.4$$

$$\text{When } x = 2.1 \qquad \frac{dy}{dx} = 4(2.1) - 8 = +0.4$$

We have, therefore, a minimum point where $x = 2$.

If it is desired to *evaluate* the minimum value of y at this point, we put $x = 2$ in (1), getting

$$y_{\text{min}} = 2(2)^2 - 8(2) - 4 = -12$$

That is, y has a minimum value at the point $(2, -12)$.

The above form of test may be called the *near-point test.* Remember that we must test with values of x very close to the value which gives a zero derivative, since some functions go through complicated changes in short intervals.

Sometimes we do not need actually to test the points located by setting the derivative equal to zero, because our knowledge of a physical problem often tells us whether maxima or minima must exist at such points (see Sec. 8-5).

Example. If $y = 3x^4 - 8x^3 + 6x^2$, locate any maximum and minimum values of y.

Differentiating,

$$\frac{dy}{dx} = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1) = 12x(x - 1)^2$$

Equating this to zero and solving for x,

$$12x(x - 1)^2 = 0$$
$$x = 0, +1, +1$$

Testing the point where $x = 0$,

$$\text{When } x = -0.1 \qquad \frac{dy}{dx} = 12(-0.1)^3 - 24(-0.1)^2 + 12(-0.1) = -1.452$$

$$\text{When } x = +0.1 \qquad \frac{dy}{dx} = 12(0.1)^3 - 24(0.1)^2 + 12(0.1) = +0.972$$

Therefore a minimum value of y exists where $x = 0$. Next, we test the point where $x = 1$:

$$\text{When } x = 0.9 \qquad \frac{dy}{dx} = 12(0.9)^3 - 24(0.9)^2 + 12(0.9) = +0.108$$

$$\text{When } x = 1.1 \qquad \frac{dy}{dx} = 12(1.1)^3 - 24(1.1)^2 + 12(1.1) = +0.132$$

Hence, there is neither a maximum nor a minimum value of y where $x = 1$ but a point of inflection instead.

QUESTIONS

1. What is the meaning of a statement that "a function y has a maximum value where $x = 4$"?
2. What do we mean by the *absolute maximum* value reached by a function? The *absolute minimum*?
3. What meaning does the term *ordinary maximum* convey? *Ordinary minimum?*
4. What are three possible conditions under which a graph might have a horizontal tangent line?
5. What is the *error* in a statement that "if, for a given value of x, the graph of a function y has a horizontal tangent line, then there must be a maximum or a minimum value of y for that value of x"?
6. What method is used for locating test points for maxima or minima?
7. Describe the near-point test for a maximum or minimum point.
8. Having definitely established that a maximum (or a minimum) value of a function $y = f(x)$ occurs for a certain value of x, how would you find the actual maximum (or minimum) value of y?

PROBLEMS

In these problems *locate* and *evaluate* any ordinary maxima or minima. State locations of any points of inflection you can find.

1. $y = x^3$
2. $y = x^4$
3. $y = 2x^2 - 6x + 5$
4. $y = 12x - x^2$
5. $y = x^2 - 8x$
6. $y = 12x - x^3$
7. $y = 2 + 6x - 3x^2$
8. $y = 4x^3 - x^4$
9. $y = x^3 - 9x^2 + 27x + 11$
10. $y = (x - 1)^4$
11. $y = x^3 - 3x^2 - 9x + 3$
12. $y = x^3 - 5x^2 + 3x$
13. $y = x^4 + 6{,}000$
14. $y = x^4 - x^3/3$
15. $y = 2x^4 - 16x^2 + 20$

8-4 The second-derivative test. In Fig. 8-3, we see that the graph of the function y rises to an ordinary maximum at point A, then falls. Accordingly, the slope of a tangent line is first positive, as for T_1; then zero, as for T_2, and finally negative, as for T_3. That is, the *slope* of the tangent line is continually decreasing as we go past the maximum point. Another way of saying this is to state that the derivative of y with respect to x is decreasing, or that the second derivative of y with respect to x is negative at the maximum point.

Similarly, Fig. 8-4 shows a graph having an ordinary minimum point at B. The slope of the tangent line (and accordingly the value of the derivative) goes from negative, through zero, to positive as we go past B. That is, the second derivative of y with respect to x is positive at the minimum point.

The above facts lead us to the following *second-derivative test* for maxima and minima:

1. If, for a given value of x, the first derivative dy/dx is zero and the second derivative d^2y/dx^2 is negative, then an ordinary maximum of y occurs at that value of x.

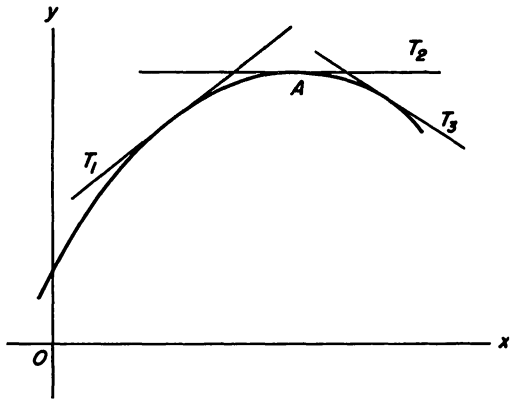


Fig. 8-3

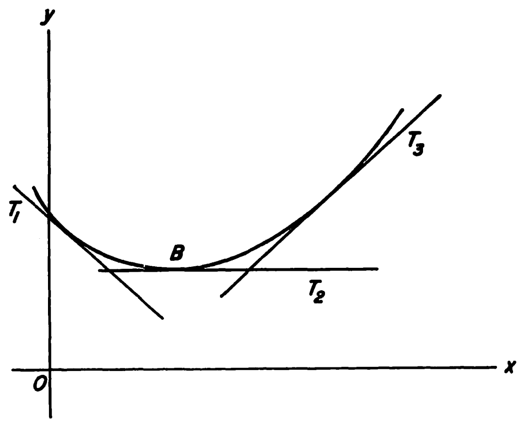


Fig. 8-4

2. If, for a given value of x, the first derivative dy/dx is zero and the second derivative d^2y/dx^2 is positive, then an ordinary minimum of y occurs for that value of x.

3. If, however, both the first and second derivatives are equal to zero, *the test fails*, and we may have either a maximum, a minimum, or a point of inflection. In this case, the near-point test should be applied, for this is a sure test.

Example 1. Locate any maxima or minima of the function $y = x^2 - 3x^3$.
Differentiating once,

$$\frac{dy}{dx} = 2x - 9x^2 = x(2 - 9x) \tag{A}$$

Setting this equal to zero and solving for x,

$$x(2 - 9x) = 0$$
$$x = 0, \tfrac{2}{9}$$

Differentiating Eq. (A),

$$\frac{d^2y}{dx^2} = 2 - 18x$$

When $x = 0$, this is positive, so that y goes through a minimum where $x = 0$. When $x = \frac{2}{9}$, the second derivative is negative, so that y has a maximum where $x = \frac{2}{9}$.

Example 2. Locate the maxima and minima, if any, of the function $y = 5x^2 - 20x$.

Differentiating,

$$\frac{dy}{dx} = 10x - 20 = 10(x - 2) \tag{A}$$

Equating to zero,

$$10(x - 2) = 0$$
$$x = 2$$

Differentiating Eq. (A),

$$\frac{d^2y}{dx^2} = 10$$

Since the second derivative here is *always* positive, the function may have ordinary minima but no ordinary maxima or other horizontal-tangent points. Thus, there is a minimum of y where $x = 2$.

Example 3. If $y = 2x^3 - 12x^2 + 24x + 7$, find any ordinary maxima or minima of this function.

We get

$$\frac{dy}{dx} = 6x^2 - 24x + 24$$
$$6(x^2 - 4x + 4) = 6(x - 2)^2 = 0$$
$$x = 2, 2$$

The second derivative is

$$\frac{d^2y}{dx^2} = 12x - 24$$

When $x = 2$, the second derivative becomes zero, so that the second-derivative test fails. Resorting to the near-point test, we find that there is a point of inflection where $x = 2$.

QUESTIONS

1. Describe the second-derivative test for ordinary maxima or minima.
2. State the procedure to be followed in testing for a maximum or minumum at a point where both the first and the second derivatives are equal to zero.

PROBLEMS

In the following problems find the values of x for which y is a maximum or a minimum (state which). Use the second-derivative test where applicable. (This test may also be tried on examples from the group following Sec. 8-3.)

1. $y = x^2$
2. $y = 2x - x^2$
3. $y = (x + 1)^2$
4. $y = 4x^2 - 4x$
5. $y = 2x^3 + 3x^2 - 2$
6. $y = x^3 - 3x^2 + 6$
7. $y = (x - 1)^3$
8. $y = (1 - x^2)^2$
9. $y = x^3 - 6x^2 + 9x + 31$
10. $y = x^4 - 8x^2 + 13$

8-5 Applications of maxima and minima. You should study the following examples carefully to familiarize yourself with the methods involved.

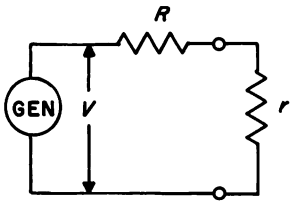


Fig. 8-5

Example 1. A dc generator (Fig. 8-5) has an internal resistance of R ohms and develops an open-circuit emf of V volts. (*a*) Show that the generator delivers the greatest possible amount of power to a load when the load resistance r is made equal to R. (*b*) Find a formula for the greatest power which the generator can supply, that is, its power into a matched load.

The current supplied by the generator is

$$i = \frac{V}{R + r}$$

The power in the load will be

$$p = i^2 r = \left(\frac{V}{R + r}\right)^2 r = \frac{V^2 r}{R^2 + 2Rr + r^2}$$

Differentiating and setting the derivative equal to zero,

$$\frac{dp}{dr} = \frac{V^2(R^2 - r^2)}{(R+r)^4}$$

$$V^2(R^2 - r^2) = 0$$

$$r = \pm R$$

Since only a positive value of load resistance can be obtained physically, we have

$$r = R$$

This is a possible value of r for a maximum of p. Using the second-derivative test,

$$\frac{d^2p}{dr^2} = -2V^2r$$

Therefore a maximum load power p is obtained when $r = R$. This completes part (*a*) of the problem. To answer part (*b*), we put $r = R$ in our *power* equation above, and find that the greatest power which can be delivered by the generator is

$$p_{\max} = \frac{V^2}{4R} \qquad (4)$$

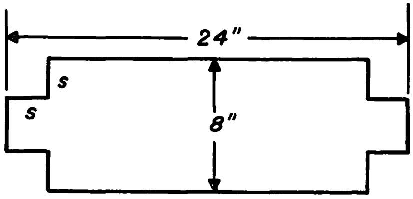


Fig. 8-6

Example 2. A sheet of metal 8×24 inches is to be used for constructing a radio chassis. A square piece is to be cut from each corner, and the sides are to be turned down to form the chassis. How large should each corner square be cut so that the chassis will contain a maximum volume in cubic inches?

As seen in Fig. 8-6, the top of the chassis will have an area $A = (24 - 2s)(8 - 2s) = 4s^2 - 64s + 192$ square inches. The volume is then

$$V = As = 4s^3 - 64s^2 + 192s \qquad \text{cubic inches}$$

Differentiating and equating the derivative to zero,

$$\frac{dV}{ds} = 12s^2 - 128s + 192$$

$$4(3s^2 - 32s + 48) = 0$$

Solving for s by the quadratic formula,

$$s = 8.86 \text{ or } 1.81$$

The first result is obviously too large from a physical viewpoint. Applying the second-derivative test to the second value of s,

$$\frac{d^2V}{ds^2} = 24s - 128$$

This is negative when $s = 1.81$, so that a maximum volume V occurs when $s = 1.81$. Thus we should cut squares from the sheet which are 1.81 inches on each side.

Example 3. The mutual inductance between two windings is 2 henrys. If the current through one of the windings is $i_1 = 3t^2 - t^3$ amperes, find (*a*) at what time the induced emf v_2 in the second winding has its greatest *negative* value, (*b*) the value of this peak of voltage, and (*c*) the amount of current i_1 when the induced-voltage peak occurs.

We write

$$v_2 = -M\frac{di_1}{dt} = -2(6t - 3t^2) = 6t^2 - 12t \qquad \text{volts}$$

Differentiating and setting the derivative equal to zero,

$$\frac{dv_2}{dt} = 12t - 12 = 12(t - 1) \qquad \text{volts per second}$$

$$12(t - 1) = 0$$

$$t = 1 \text{ second}$$

The second-derivative test gives $d^2v_2/dt^2 = 12$, which is positive for all values of t; hence a minimum of v_2 (maximum negative value) occurs when $t = 1$. This answers part (*a*) of the problem. Part (*b*) is solved by letting $t = 1$ in the expression above for v_2, getting

$$v_{2,\,\min} = 6(1)^2 - 12(1) = -6 \text{ volts}$$

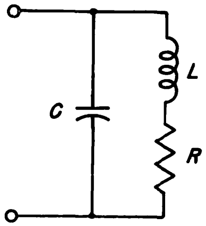


Fig. 8-7

Part (*c*) is solved by letting $t = 1$ in the given formula for i_1:

$$i_1 = 3(1)^2 - (1)^3 = 2 \text{ amperes}$$

Example 4. A series RL combination is connected in parallel with a capacitance C (Fig. 8-7). It can be shown that at an angular frequency $\omega = 2\pi f$, the impedance of the circuit is

$$|Z| = (R^2 + \omega^2L^2)^{1/2}[\omega^2C^2R^2 + (\omega^2LC - 1)^2]^{-1/2} \qquad \text{ohms}$$

If L and R are fixed, what value of C causes $|Z|$ to be a maximum?

Differentiating,

$$\frac{d|Z|}{dC} = -(R^2 + \omega^2L^2)^{1/2}[\omega^2C^2R^2 + (\omega^2LC - 1)^2]^{-3/2}(\omega^2R^2C + \omega^4L^2C - \omega\, L)$$

Setting this equal to zero, and dividing by

$$-\omega^2(R^2 + \omega^2L^2)^{1/2}[\omega^2C^2R^2 + (\omega^2LC - 1)^2]^{-3/2}$$

we find

$$R^2C + \omega^2L^2C - L = 0$$

$$C = \frac{L}{R^2 + \omega^2L^2} \qquad \text{farads}$$

This could presumably be tested to disclose whether or not it actually produces a maximum of $|Z|$. But such a test would be tedious, and our knowledge of tuned circuits tells us that, as C is varied, the impedance graph has a definite maximum but no minima or points of inflection.

Study of the preceding examples discloses the following procedure in solving problems in maxima and minima:

1. Wherever possible, *make a sketch* illustrating the problem. Assign symbols to the quantities involved.

2. *Study the problem* carefully. Observe how the various quantities are related to each other. Note specifically which variable is to be maximized or minimized.

3. *Write an equation* for the quantity which is to be maximized or minimized. This should be expressed as a function of just one independent variable. In preparing this equation, it is often necessary to write first some other equations relating the quantities in the problem, obtained from a knowledge of the subject matter of the problem. These relations may permit substitutions to be made so that the number of variables is reduced.

4. *Differentiate* the equation for the quantity to be maximized or minimized.

5. *Set the derivative just obtained equal to zero,* and find the corresponding values of the independent variable. This locates possible maxima, minima, and points of inflection.

6. *Test the values of the independent variable,* found in step 5, or *apply knowledge of conditions of the problem,* to establish which values represent actual maxima or minima.

7. If it is desired to *evaluate* the maxima or minima, this may be done by substituting into the equation of step 3 the values of the independent variable found in step 6 to represent actual maxima or minima.

QUESTION

1. Describe the steps to be followed in solving a practical problem requiring the finding of a maximum or a minimum point.

PROBLEMS

In the following problems apply in each case a suitable test or explain what physical reasoning was employed to establish the existence of the maxima or minima.

1. What maximum height is attained by a projectile fired vertically upward so that its height h feet is $h = 1{,}024t - 16t^2$?

2. A radio chassis is to be made as in Example 2, Sec. 8-5, from a sheet of metal 9×20 inches. Find the side s of the square to be cut from each corner to make the enclosed volume a maximum.

3. During a certain interval the current in a resistor of R ohms is $i = t^3/3 - t$ amperes. When does the minimum current flow in the resistor? (NOTE: A *minimum current* may be an actual maximum value in a direction opposed to that assumed to be positive.)

4. An antenna wire is suspended so that its height h feet varies with distance s feet from one end according to $h = 0.0004s^2 - 0.04s + 67$. What is the height of the lowest point?

5. The work done by a solenoid in moving an armature varied according to $w = 2t^3 - t^4$ joules. What was the greatest power developed by the solenoid?

6. The strength of a timber of rectangular cross section can be shown to be $S = kbd^2$, where k is a constant, b the breadth of the timber, and d its depth. If, in the construction of a transmitting-plant building, a timber is to be cut from a log of diameter a, find the dimensions of the beam having the greatest strength.

7. A rectangular apparatus box is to have a square bottom and is to contain 650

cubic inches. Find its dimensions so that the least metal will be used for the box, including the cover. (Neglect seams and waste.)

8. Same as Prob. 7, but omit the cover of the box.

9. Transformer oil is to be sealed in a cylindrical metal container, having a volume of 2,000 cubic inches. Find the radius r and the height h of the container requiring the least metal. (Neglect seams and waste.)

10. What is the greatest current in an 8-microfarad capacitor if the applied voltage is $v = 250t^2 - 200t^3$ volts?

11. The charge transmitted through a circuit varied according to $q = 4t^4 - t^3$ coulombs. After what time was the current a maximum in the *negative* direction?

$V_{ind} = -L\frac{di}{dt}$

12. If the circuit of Prob. 11 included a 16-henry inductor, what was the greatest *positive* voltage induced in it?

13. After what time is the power in the resistor of Prob. 3 a maximum?

14. If R and C are kept fixed in the circuit of Example 4, Sec. 8-5, what L gives a maximum $|Z|$?

15. A voltage V is applied to the primary of a transformer. The primary circuit resistance and reactance are R_p and X_p, and the secondary circuit values are R_s and X_s. A mutual inductance M exists between the windings. It can be shown that the secondary current is

$$|I_2| = \frac{V\omega M}{[(X_pR_s + X_sR_p)^2 + (R_pR_s - X_pX_s + \omega^2M^2)^2]^{1/2}}$$

What X_s makes $|I_2|$ a maximum, other values remaining constant? (HINT: Since the numerator in this expression is made up of constants, we may simply make the square of the denominator a minimum.)

16. An electronics factory makes u devices a month, which it sells for x dollars each. If the monthly expense of the factory varies with production according to $A = 12{,}500 + 32u - 0.03u^2$, and if the number of units produced and sold varies with the price according to $u = 850 - 6x$, what selling price provides the greatest profit P?

8-6 Conclusion. It should be pointed out in conclusion that proofs of many of the results presented in the foregoing paragraphs are quite complicated, and are best reserved for more advanced courses.

REFERENCES

1. H. M. BACON: "Differential and Integral Calculus," 2d ed., pp. 71–89, McGraw-Hill Book Company, Inc., New York, 1955.
2. F. L. GRIFFIN: "Mathematical Analysis: Higher Course," pp. 58–59, Houghton Mifflin Company, Boston, 1927.
3. C. R. WYLIE: "Calculus," pp. 111–113, McGraw-Hill Book Company, Inc., New York, 1953.

9

Integrals

Up to this point we have been concerned chiefly with the differentiation of functions and related problems. We now turn to a process which may be thought of as the *reverse* of differentiation.

9-1 Integration. Consider the problem

➢ Given the differential (or derivative) of a function, how can we find the function?

For example, if

$$dy = x^3\,dx \tag{1}$$

what must be the expression for y?

[The problem might also be stated:

➢ Given the rate of change of a function, find the function.

For Eq. (1) is equivalent to the rate expression

$$\frac{dy}{dx} = x^3 \tag{2}$$

and it remains to find what function y has the derivative (2).]

We proceed to solve the problem, that is, to find y. From our knowledge of differentiation, we assume that y must be some *power function*, that is, a function of the form $y = bx^m$, where b and m are constants. And m must be equal to 4, since in differentiating a power function, we get an exponent *one less* than that in the original function. But the original function here cannot be simply

$$y = x^4 \tag{3}$$

for differentiation of this would give $dy = 4x^3\,dx$, which is four times as large as the expression (1). Accordingly, we try multiplying the right member of (3) by $\frac{1}{4}$:

$$y = \tfrac{1}{4}x^4 \tag{4}$$

and this can be differentiated to give (1). Thus $y = \frac{1}{4}x^4$ is a solution of our problem.

However, the function $y = \frac{1}{4}x^4$ is not the only one which can be differentiated to give (1). For if

$$y = \tfrac{1}{4}x^4 + 3$$

or if

$$y = \tfrac{1}{4}x^4 - 10{,}000$$

or, in fact, if any constant, positive or negative, is added to (4), the result will still differentiate to give (1). This added constant may be in any form. The following functions, for example, will also differentiate to give (1):

$$y = \tfrac{1}{4}x^4 + 10^{-4}$$
$$y = \tfrac{1}{4}x^4 - \sin 120°$$
$$y = \tfrac{1}{4}x^4 + \log 2 \qquad \text{etc.}$$

We say, then, that

$$\text{if} \qquad dy = x^3\,dx \qquad \text{then} \qquad y = \tfrac{1}{4}x^4 + C \tag{5}$$

where C is any constant.

The process of finding a function which has a given differential (or derivative) is called *integration*.

This may be considered the reverse process of differentiation.

Instead of the form (5), we customarily write

$$\int x^3\,dx = \tfrac{1}{4}x^4 + C \tag{6}$$

The symbol $\int$ is called the *integral sign*. The left member of (6) is called the *integral of $x^3\,dx$* or the *integral of x^3 with respect to x*. The function x^3 is called the *integrand*. The constant C in the right member is called the *constant of integration*. (Instead of the commonly used C, in this book we shall often use other symbols, like K, because C will often represent capacitance.)

Despite the terminology just presented, we must keep in mind that (6) has precisely the same meaning as (5).

9-2 Integration formulas. To simplify our work, we derive some formulas for finding certain integrals. The examples must be studied carefully.

a. The integral $\int u^n\,du$. The first formula which we consider is

will not work

$$\boxed{\int u^n\,du = \frac{u^{n+1}}{n+1} + C} \qquad \text{when } n=-1 \tag{7}$$

We prove its correctness by letting the right member equal y, and differentiating:

$$y = \frac{1}{n+1}u^{n+1} + C$$

$$\frac{dy}{du} = \frac{n+1}{n+1}u^n = u^n \qquad \text{or} \qquad dy = u^n\,du$$

That is, the differential of the right member of (7) is identical to the quantity under the integral sign in the left member. Therefore our integration is correct.

Incidentally, we shall henceforth use this procedure of *differentiating* the result of an integration to check its correctness.

Example 1. Find $\int x^5\,dx$.
Applying (7), we see that $u = x$, $n = 5$, $du = dx$.
This gives

$$\int x^5\,dx = \tfrac{1}{6}x^6 + C$$

Checking this result by differentiation,

$$\frac{d}{dx}\left(\frac{1}{6}x^6 + C\right) = x^5$$

$$d\left(\tfrac{1}{6}x^6 + C\right) = x^5\,dx$$

Therefore the integration was correctly performed.

CAUTION: Formula (7) does not work if the given exponent n is equal to -1. In this case, the formula gives meaningless results, as you should verify. We must await a later chapter to get a formula for $\int u^{-1}\,du$ (or, as it is usually written, $\int du/u$). We say, therefore, that

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1 \tag{7a}$$

It is recalled (Sec. 6-1) that a differential is not always a simple form like du or dx but might be an expression like, say, $3x^2\,dx$. This fact helps us to integrate some more complicated expressions.

Example 2. Find $\int 3x^2(x^3 + 2)^3\,dx$.

Here, let $u = x^3 + 2$. Then du must be equal to $3x^2\,dx$. We may rewrite the given integral

$$\int (x^3 + 2)^3 3x^2\,dx$$

where the expression outside the parentheses makes up the *differential* of that within the parentheses. The given integral then is of the form $\int u^n\,du$, where $u = x^3 + 2$, $n = 3$, $du = 3x^2\,dx$. Applying (7*a*),

$$\int (x^3 + 2)^3 3x^2\,dx = \tfrac{1}{4}(x^3 + 2)^4 + C$$

Checking by differentiation,

$$d[\tfrac{1}{4}(x^3 + 2)^4 + C] = (x^3 + 2)^3 3x^2\,dx$$

Therefore the result is correct. (In an integral which looks like the one given above we have no guarantee that an expression outside the parentheses is necessarily the differential of that within the parentheses, but this possibility is always worth checking. Formulas to be developed later are helpful in many other cases.)

b. The integral $\int du$. The form $\int du$ may be written $\int u^0\,du$. Applying (7*a*),

$$\int du = u + C \tag{8}$$

This is readily checked by differentiation. Similarly,

$$\begin{aligned} \int dx &= x + C \\ \int d(\cos^2\theta) &= \cos^2\theta + C \\ \int d(uv) &= uv + C \qquad \text{etc.} \end{aligned}$$

c. Effect of a constant multiplier. Our next integration formula states that

$$\int a\,du = au + C \tag{9}$$

where a is a given constant. This may be checked by differentiating the right member with respect to u:

$$d(au + C) = a\,du$$

which is the given differential.

Note that the right member of (9) is a times as great as that obtained in Formula (8) for $\int du$ (except for C, which can actually represent *any* constant in either formula).

A result of (9) is that

Multiplying any differential by a constant simply multiplies its integral by the same constant.

Stated in another way, a *constant* multiplier may be moved at will from one side of the integral sign to the other:

$$\int a\,du = a\int du \tag{9a}$$

Example 3. Find $\int 5x^3\,dx$.

This may be written $5\int x^3\,dx$, which gives

$$5 \times \tfrac{1}{4}x^4 + C \qquad \text{or} \qquad \tfrac{5}{4}x^4 + C$$

This checks by differentiation.

Example 4. Evaluate $\int (x^4 + 7)^2 x^3\,dx$.

We note that, if the quantity $x^3\,dx$ were multiplied by 4, it would become the differential of the quantity within the parentheses. Since the missing factor 4 is a *constant*, we proceed to multiply $x^3\,dx$ by 4; then we multiply by $\frac{1}{4}$ outside the integral sign, to avoid changing the value of the result:

$$\int (x^4 + 7)^2 x^3\,dx = \tfrac{1}{4}\int (x^4 + 7)^2 4x^3\,dx$$

This is now of the form $\int u^n\,du$, where $u = x^4 + 7$, $n = 2$, $du = 4x^3\,dx$. Therefore,

$$\int (x^4 + 7)^2 x^3\,dx = \tfrac{1}{4} \times \tfrac{1}{3}(x^4 + 7)^3 + C = \tfrac{1}{12}(x^4 + 7)^3 + C$$

This checks by differentiation, yielding the given differential $(x^4 + 7)^2 x^3\,dx$.

Example 5. Find $\int (3x^2 + x^3)^{2/3}(2x + x^2)\,dx$.

We observe that, if $u = 3x^2 + x^3$, then $du = (6x + 3x^2)\,dx$. If we supply a multiplier of 3, the quantity $(2x + x^2)\,dx$ will take the value of this differential. Accordingly we multiply by 3 and by $\frac{1}{3}$:

$$\begin{aligned}\int (3x^2 + x^3)^{2/3}(2x + x^2)\,dx &= \tfrac{1}{3}\int (3x^2 + x^3)^{2/3}(6x + 3x^2)\,dx \\ &= \tfrac{1}{3} \times \tfrac{3}{5}(3x^2 + x^3)^{5/3} + C = \tfrac{1}{5}(3x^2 + x^3)^{5/3} + C\end{aligned}$$

This checks by differentiation.

It must be remembered that only *constants* can be supplied in the manner of Examples 4 and 5. For instance, the integral $\int (x^2 + 10)^{1/2}\,dx$ cannot be found by any method we have learned thus far. True, *if* we could legitimately supply the factor $2x$ (and divide by $2x$ outside the integral sign), we could then use the formula for $\int u^n\,du$. But $2x$ involves a variable, so that this operation would give an *incorrect* result.

d. Integral of a sum. From our knowledge of differentiation,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

This gives

$$du + dv = d(u + v)$$

Integrating each member,

$$\int (du + dv) = \int d(u + v)$$

By (8), the right member yields $(u + v) + C$, so that

$$\int (du + dv) = u + v + C \tag{10}$$

We also note that

$$\int du + \int dv = u + v + C \tag{11}$$

Comparing with (10), we see that

$$\int (du + dv) = \int du + \int dv \tag{12}$$

Thus,

The integral of the sum of two or more differentials is equal to the sum of the integrals of the separate differentials.

Example 6. Find $\int (x^2 + 2x^3)\, dx$.

Mentally, we note that this is the integral of the sum of two differentials, $\int (x^2\, dx + 2x^3\, dx)$. Applying (12), we write

$$\int (x^2 + 2x^3)\, dx = \frac{x^3}{3} + \frac{x^4}{2} + C$$

where again the single constant C can be considered the sum of the constants of integration associated with the terms of the given integral. The result can be checked by differentiation.

9-3 Suggestions for working integration problems. The following ideas should be kept in mind while working problems in integration:

1. Remember that it is a *differential* which we are integrating. Therefore, there *must* be a differential factor like dy, dx, or dz, etc., somewhere following the $\int$ sign. (Needless to say, it must be the correct factor for the particular problem.) Anyone who incorrectly writes $\int 5x^7$ with no differential expression is displaying improper thinking and likely does not understand what he is doing, even though he might get some correct answers.

2. Watch the parentheses in integral expressions involving sums. If we mean $\int (x^2 + 2x)\, dx$, we do *not* write $\int x^2 + 2x\, dx$. The latter expression is meaningless.

3. When writing integrals of the form discussed in this chapter, always include the $+\ C$, or constant of integration, in the result.

4. Form the habit of testing the answers to integration problems by differentiating back.

QUESTIONS

1. Define the term *integration*.
2. State the formula for finding the integral $\int u^n\, du$. For what value(s) of n does this formula *not* apply?
3. What is the value of $\int du$?
4. If a differential is multiplied by a constant, what is the effect upon its integral?
5. How do we find the integral of the sum of two differentials?
6. Find the *errors* in the following expressions: (*a*) $\int x^3 - x\, dx$, (*b*) $\int x^6$, (*c*) $\int dy = y$, (*d*) $\int x^2 = x^3/3$.
7. When writing the result of an integration, why is it necessary to supply an *arbitrary constant of integration*?

8. How do we check the correctness of an integration?

9. An instructor prepared an examination in integration. Each question gave a function to be integrated, and the answer consisted of selecting the correct result from several given choices. Explain why this was not a good test of a student's ability in integration.

PROBLEMS

Know All 1-36

Carry out the following integrations. Test the results by differentiation.

1. $\int dr$
2. $\int dt$
3. $\int 3dx$
4. $\int 2\pi\, d\theta$
5. $\int a\, dx$
6. $\int \frac{5}{2}\, dx$
7. $\int y^2\, dy$
8. $\int 2x\, dx$
9. $\int a\phi^3\, d\phi$
10. $\int 3w^2\, dw$
11. $\int 12s^3\, ds$
12. $\displaystyle\int \frac{5\pi}{2}\phi^4\, d\phi$
13. $\int 2y^7\, dy$
14. $\int 5r^3\, dr$
15. $\int \sqrt{x}\, dx$
16. $\int \sqrt[3]{y}\, dy$
17. $\displaystyle\int \frac{dt}{t^6}$
18. $\int 3x^{5/2}\, dx$
19. $\int 3y^8\, dy$
20. $\int (3 - x)\, dx$
21. $\int (75 - 4x^3)\, dx$
22. $\int (2x^3 - 5x)\, dx$
23. $\int (x^2 - \sqrt{x})\, dx$
24. $\int (3z^2 + 3z^5)\, dz$
25. $\int (\sqrt[3]{s} - s)\, ds$
26. $\int (x^2 + 5)2x\, dx$
27. $\int 15z^2(5z^3 + 20)\, dz$
28. $\int 5t^4(2 - t^5)^5\, dt$
29. $\int t\sqrt{2t^2 + 1}\, dt$
30. $\displaystyle\int \frac{y\, dy}{(y^2 + 100)^2}$
31. $\displaystyle\int \frac{r^3\, dr}{\sqrt{200r^4 + 2}}$
32. $\displaystyle\int \frac{\theta^2\, d\theta}{\sqrt{\pi - \theta^3}}$
33. $\int 6(2t^3 + t^6)^4(t^2 + t^5)\, dt$
34. $\int (t^3 - 3t)^3(t^2 - 1)\, dt$
35. $\displaystyle\int \frac{(y^3 + 1)\, dy}{\sqrt{20y + 5y^4}}$
36. $\displaystyle\int \frac{y(10 - y)\, dy}{\sqrt[3]{y^3 - 15y^2}}$

9-4 Constant of integration. We have seen that the integral of any given differential includes, in general, an arbitrary constant which may have any value, either positive or negative. For this reason, integrals of the form treated in this chapter are called *indefinite integrals*, but we must not get a notion that such integrals are not of practical value. As we shall see, quite the opposite is true.

Suppose that we are given a certain differential and that we are able to obtain a formula for its integral. What *additional* information do we need to establish definitely the value of the constant of integration for this particular problem?

To answer this question let us take as an example the differential

$$dy = 2x\, dx \tag{13}$$

This can be integrated to give

$$y = x^2 + C \tag{14}$$

Several graphs are shown in Fig. 9-1, each of which satisfies the form (14). These graphs have the same size and shape, but they have different heights; that is, the value of y for any given value of x differs from one

graph to another only in the value assigned to C. An *infinite* number of such graphs could be drawn, each satisfying (14).

Now suppose that we not only know that $dy = 2x\,dx$, but let it also be given that

$$\text{When } x = 0 \qquad y = 2 \tag{15}$$

This second bit of information establishes that the particular graph which fits our problem goes through the point (0,2), and this immediately

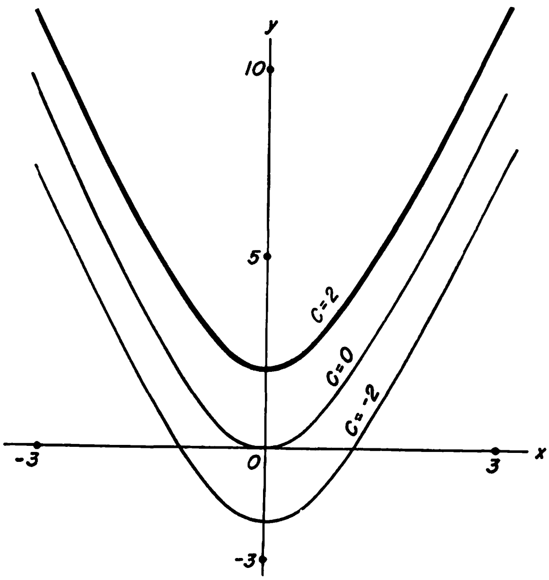


Fig. 9-1

establishes the graph drawn in a heavy line as the one which applies to the problem. That is, it determines the value of C. For if we substitute the values $x = 0$ and $y = 2$ in (14), we get

$$2 = (0)^2 + C \qquad \text{or} \qquad C = 2 \tag{16}$$

If we substitute (16) into (14), we get

$$y = x^2 + 2 \tag{17}$$

And (17) not only satisfies (13), but it satisfies the conditions imposed by (15) as well.

We see then that

➢ In order to get a complete formula for an integral which satisfies a particular problem, we must be given not only the formula for the differential which applies, but also one additional bit of information:

we must know *one point* on the graph representing the desired formula.

In practical problems the point which we are given will often (but not always) be that where the independent variable x is equal to zero; that is, we shall be told what is the value of y when x is zero, as in the example above. In such cases, we say that we are given the *initial conditions* (or *boundary values*) of the problem.

Evaluation of the constant of integration is illustrated for practical cases in Sec. 9-6.

9-5 Equations solved by integration. Let it be given that the derivative of some function $y = f(x)$ is dy/dx:

$$\frac{dy}{dx} = f'(x)$$

Since the two members of the above equation are equal, their integrals with respect to x must be equal:

$$\int \frac{dy}{dx}\,dx = \int f'(x)\,dx$$

The left member becomes

$$\int dy = y + C_1$$

So far we have not been given the form of the function $f'(x)$ in terms of x, so we can only indicate it by the symbols already shown. Let us, however, include its associated constant of integration, C_2. We have then

$$\begin{aligned} y + C_1 &= \int f'(x)\,dx + C_2 \\ y &= \int f'(x)\,dx + C_2 - C_1 \end{aligned}$$

We may represent the constant difference $C_2 - C_1$ by the symbol C:

$$y = \int f'(x) + C$$

The above operations illustrate that, for brevity, we can solve a relation like

$$\frac{dy}{dx} = f'(x)$$

to find y, by the following steps:

1. Multiplying both members by the denominator differential in the left member.
2. Writing integral signs before the two resulting differentials.
3. And completing the indicated integrations, including only a *single* constant of integration.

Example. Given that $dy/dx = 8x^3$, find y.
We write

$$\frac{dy}{dx} = 8x^3$$
$$dy = 8x^3\,dx$$
$$\int dy = 8\int x^3\,dx$$
$$y = 2x^4 + C$$

9-6 Applications of integrals; the rate problem reversed. We have been treating integration as the finding of a function which has a given differential. But we have also seen [Eq. (2)] that this problem can be interpreted as the finding of a function which has a given rate of change. We shall now consider some uses of this interpretation.

a. Finding distance from speed. Let an object move a distance s along a straight line. Let its speed $\mathbf{v}$ vary as a function of time t. We may write

$$\frac{ds}{dt} = \mathbf{v} \qquad \text{or} \qquad \int ds = \int \mathbf{v}\,dt \tag{18}$$

$$s = \int \mathbf{v}\,dt \tag{19}$$

As soon as we are given a formula for $\mathbf{v}$ as a function of t, we may proceed to integrate the right member of (19), getting a formula for the distance s traveled by the object in time t. And when we write the value of the integral of the right member, we shall include the single required constant of integration.

Example 1. A falling object moved at a speed $\mathbf{v} = 32t$ feet per second. If the object started from rest when $t = 0$, find the distance it fell in 4 seconds.

Applying (19),

$$s = \int \mathbf{v}\,dt = \int 32t\,dt = 16t^2 + C \tag{A}$$

To evaluate C, note that the object started from rest; that is, the distance which it had traveled when $t = 0$ was zero. We may then substitute $s = 0$, $t = 0$ in Equation (A) to show the conditions when the fall began:

$$0 = 16(0)^2 + C \qquad \text{or} \qquad C = 0$$

Therefore (A) becomes

$$s = 16t^2 \qquad \text{feet}$$

Substituting $t = 4$,

$$s = 16(4)^2 = 256 \text{ feet}$$

b. Finding speed from acceleration. It is recalled that the acceleration of an object moving along a straight path is given at any time t by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

where $\mathbf{v}$ is the speed of the object. This can be written

$$d\mathbf{v} = \mathbf{a}\,dt$$

which integrates to give

$$\mathbf{v} = \int \mathbf{a}\,dt \tag{20}$$

Example 2. Over a portion of its path of operation, a rod is actuated by a motor so that its acceleration is $42t$ meters per second per second. If it starts from rest, when $t = 0$, what is the speed when $t = 0.25$ second?

Applying (20),

$$\mathbf{v} = 42\int t\,dt = 21t^2 + C \qquad \text{meters per second}$$

Since $\mathbf{v} = 0$ when $t = 0$,

$$0 = 21(0)^2 + C \qquad \text{or} \qquad C = 0$$

Therefore $\qquad \mathbf{v} = 21t^2 \qquad$ meters per second

Substituting $t = 0.25$,

$$\mathbf{v} = 21(0.25)^2 = 1.31 \text{ meters per second}$$

Example 3. An airplane fires a projectile vertically upward from an elevation of 4,000 feet. If the initial upward speed of the projectile is 960 feet per second, and if the downward acceleration (negative upward acceleration) due to gravity is 32 feet per second per second, find (*a*) the speed when $t = 20$ seconds, (*b*) the time required for the rise of the projectile to its greatest height, (*c*) the greatest height attained, (*d*) the height after 20 seconds, (*e*) the time at which the projectile strikes the earth, (*f*) the speed with which it strikes the earth. (Neglect any horizontal motion due to the motion of the plane.)

Applying (20),

$$\mathbf{v} = \int \mathbf{a}\,dt = -32\int dt = -32t + C_1 \qquad \text{feet per second}$$

Since $\mathbf{v} = 960$ when $t = 0$,

$$960 = 32(0) + C_1 \qquad \text{or} \qquad C_1 = 960$$

$$\mathbf{v} = 960 - 32t \qquad \text{feet per second} \tag{A}$$

Letting $t = 20$, we get

$$\mathbf{v} = 960 - 32(20) = 320 \text{ feet per second}$$

This answers part (*a*) of the problem. To find the time required for the projectile to reach its greatest height, we set $\mathbf{v} = dh/dt = 0$ (where we use h to indicate height):

$$960 - 32t = 0$$
$$t = 30 \text{ seconds}$$

This solves part (*b*). To get a formula for the height at any time, we apply (19) to (*A*) above:

$$h = \int \mathbf{v}\,dt = \int (960 - 32t)\,dt = 960t - 16t^2 + C_2 \qquad \text{feet}$$

Since $h = 4{,}000$ when $t = 0$,

$$4{,}000 = 960(0) - 160(0)^2 + C_2 \qquad \text{or} \qquad C_2 = 4{,}000$$
$$h = 4{,}000 + 960t - 16t^2 \qquad \text{feet} \qquad (B)$$

To find the greatest height attained, let $t = 30$ in (B):

$$h_{\max} = 4{,}000 + 960(30) - 16(30)^2 = 18{,}400 \text{ feet}$$

which answers part (c) of the problem. To solve part (d), let $t = 20$ in (B):

$$h = 4{,}000 + 960(20) - 16(20)^2 = 16{,}800 \text{ feet}$$

To find the time at which the projectile strikes the earth, let the height h be made equal to zero in (B):

$$0 = 4{,}000 + 960t - 16t^2$$
$$t^2 - 60t - 250 = 0$$

Solving by the quadratic formula,

$$t = 63.91 \text{ or } -3.91 \text{ seconds}$$

(The first of these two results is obviously the correct one.) This is the answer to part (e). To solve part (f), put $t = 63.91$ in (A):

$$\mathbf{v} = 960 - 32(63.91) = -1{,}085 \text{ feet per second}$$

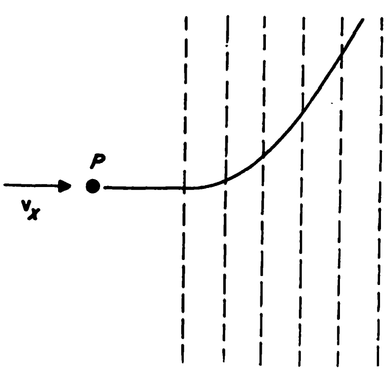


Fig. 9-2

The minus sign indicates that this is the speed in a downward direction, opposite to the direction originally taken as positive.

Example 4. A charged particle P (Fig. 9-2) has a mass m kilograms and carries a charge q coulombs. At time $t = 0$, the particle is injected at a horizontal speed $\mathbf{v}_x$ into a uniform vertical electric field of intensity $\mathbf{E}$ volts per meter. Find the equation of motion for this particle.

While the particle is in the field, its velocity has a horizontal component $\mathbf{v}_x$ and a vertical component $\mathbf{v}_y$. The horizontal distance moved by the particle in time t is

$$x = \mathbf{v}_x t \qquad \text{meters} \qquad (A)$$

Simultaneously, a force $\mathbf{F}$ kilograms in the vertical direction results from the action of the field, given by

$$\mathbf{F} = \mathbf{E}q \qquad \text{kilograms}$$

By Newton's second law, the resulting acceleration $\mathbf{a}_y$ of the particle in the vertical direction is

$$\mathbf{a}_y = \frac{d\mathbf{v}_y}{dt} = \frac{\mathbf{F}}{m} = \frac{\mathbf{E}q}{m} \qquad \text{meters per second per second}$$

Rewriting this equation and integrating,

$$\mathbf{v}_y = \frac{\mathbf{E}q}{m}\int dt = \frac{\mathbf{E}qt}{m} + K_1 \qquad \text{meters per second}$$

But when $t = 0$, $\mathbf{v}_y = 0$, so that $K_1 = 0$,

$$\mathbf{v}_y = \frac{\mathbf{E}qt}{m} \qquad \text{meters per second}$$

This is the rate of change of vertical distance y traveled in time t. We write

$$\frac{dy}{dt} = \frac{\mathbf{E}qt}{m} \qquad \text{meters per second}$$

Multiplying by dt and integrating,

$$y = \frac{\mathbf{E}q}{m}\int t\,dt = \frac{\mathbf{E}qt^2}{2m} + K_2 \qquad \text{meters}$$

But at time $t = 0$, the particle has moved zero distance vertically, making $K_2 = 0$:

$$y = \frac{\mathbf{E}qt^2}{2m} \qquad \text{meters} \qquad (B)$$

From (A), we substitute $t = x/\mathbf{v}_x$:

$$y = \frac{\mathbf{E}qx^2}{2m\mathbf{v}_x^2} \qquad \text{meters}$$

This is the desired equation.

c. Finding charge from current. Since the current in a circuit at any instant is defined as

$$i = \frac{dq}{dt}$$

we may write

$$dq = i\,dt$$

which, when both members are integrated, becomes

$$q = \int i\,dt \qquad \text{coulombs} \qquad \mathbf{(21)}$$

when i is in amperes and t in seconds. This conveys that we can find the total charge transmitted in a circuit by integrating the expression for the current in terms of time.

As an application of (21), we recall that the charge in a capacitor is

$$q = Cv_C \qquad \text{coulombs}$$

where C is the capacitance in farads and v_C the voltage across the capacitor terminals. Substituting (21) in this equation and rearranging, we get

$$v_C = \frac{1}{C}\int i\,dt \qquad \text{volts} \qquad \mathbf{(22)}$$

which is the formula for the voltage across a capacitor at any instant.

Example 5. A 1-microfarad capacitor is charged to 67 volts. It is then connected at an instant which we shall call $t = 0$ to a source which sends a current $i = t^2$ amperes into it. Find the voltage across the capacitor when $t = 0.1$ second.

According to (22),

$$v_C = \frac{1}{C}\int i\,dt = \frac{1}{10^{-6}}\int t^2\,dt = 3.33 \times 10^5 t^3 + K \qquad \text{volts}$$

When $t = 0$, $v_C = 67$, so that $K = 67$ volts. Then

$$v_C = 3.33 \times 10^5 t^3 + 67 \qquad \text{volts}$$

Letting $t = 0.1$ second, we get

$$v_C = 3.33 \times 10^5 \times 10^{-3} + 67 = 400 \text{ volts}$$

d. Flux or current required for a given induced emf. We have learned that when the magnetic flux ϕ linking a coil of N turns is varied, there is produced an emf in the coil given by

$$v_{ind} = -N\frac{d\phi}{dt}$$

This can be rearranged and integrated, giving

$$\phi = -\frac{1}{N}\int v_{ind}\,dt \qquad \text{webers} \tag{23}$$

If, on the other hand, the inductance of a circuit is given, rather than the number of turns, the induced emf is

$$v_{ind} = -L\frac{di}{dt}$$

Integration shows that the current required to produce a given emf is

$$i_L = -\frac{1}{L}\int v_{ind}\,dt \qquad \text{amperes} \tag{24}$$

Example 6. The current in a 10-henry inductor when $t = 0$ is 2.2 amperes. What must be the current at any time t after this instant for the induced emf to be $v_{ind} = 4.2t^2 - t$ volts?

Using (24),

$$\begin{aligned} i_L &= -\frac{1}{L}\int v_{ind}\,dt = -\frac{1}{10}\int (4.2t^2 - t)\,dt \\ &= -0.1(1.4t^3 - 0.5t^2) + K \qquad \text{amperes} \end{aligned}$$

Letting $i = 2.2$ when $t = 0$, we solve for K:

$$2.2 = -0.1[1.4(0)^3 - 0.5(0)^2] + K \qquad \text{or} \qquad K = 2.2 \text{ amperes}$$

Then

$$i = 2.2 - 0.14t^3 + 0.5t^2 \qquad \text{amperes}$$

Similarly, when the current i_1 in a winding varies, the induced emf v_2 in a second winding is given by

$$v_2 = -M \frac{di_1}{dt}$$

where M is the mutual inductance between the windings. Solving for i_1:

$$i_1 = -\frac{1}{M}\int v_2\,dt \qquad \text{amperes} \tag{25}$$

e. Finding energy (or work) from power. We have learned that the power in a circuit at any instant is

$$p = \frac{dw}{dt}$$

where w is the energy expended. This can be written $dw = p\,dt$. Upon integration this gives

$$w = \int p\,dt \qquad \text{joules} \tag{26}$$

Also, we recall that the instantaneous power in a circuit can be represented variously as $p = vi$, $p = i^2r$, or $p = v^2/r$. These expressions can be substituted for p in (26) where desired. For example, if both the voltage and the current in a circuit are known functions of time, we can get the total energy used in the circuit by

$$w = \int vi\,dt \qquad \text{joules}$$

QUESTIONS

1. If we are able to obtain a formula for an integral to within an arbitrary constant, what additional information is needed to evaluate this constant?
2. What is meant when we say that we are given the *initial values* or the *boundary conditions* in a problem?
3. Suppose an object to be moving along a straight line at a speed which is a known function of time. State a formula for the distance traveled in a given time.
4. If an object moves along a straight line with an acceleration which is a given function of time, what formula gives its speed at any time?
5. If the current in a circuit varies in a known manner with time, what formula gives the total charge transmitted through the circuit?
6. Give a formula for the voltage across a capacitor at any instant if the current i supplied to the capacitor varies in a known way with time.
7. State a formula for the magnetic flux required to induce an emf which varies in a given way with time in a coil of N turns.
8. If the inductance of a circuit is L henrys, what formula expresses the current required to induce an emf in the circuit which varies in a specified way with time?
9. If it is desired to induce a voltage v_2, which is a given function of time, in a winding which is coupled to another winding, what formula gives the current i_1 required in the latter winding?

10. If the power in a circuit changes in a known manner with time, what expression gives the energy dissipated?

11. State the formula of question 10 in terms of the instantaneous voltage and current in the circuit.

PROBLEMS

1. A projectile was fired vertically upward from ground level so that its speed was $\mathbf{v} = 1{,}280 - 32t$ feet per second. Find its height after t seconds.

2. A ball was thrown vertically upward from a point 26 feet above the earth with an initial speed of 64 feet per second. What was its greatest height? (Assume that downward acceleration due to gravity was 32 feet per second per second.)

3. If a falling object has a downward acceleration of 32 feet per second per second, show by successive integrations that (*a*) its downward speed at any time t seconds is $\mathbf{v} = 32t + \mathbf{v}_0$, where $\mathbf{v}_0$ is its initial speed; (*b*) the distance which it has traveled after t seconds is $s = 16t^2 + \mathbf{v}_0 t + s_0$, where s_0 is the distance which had been covered when we began to *count time*, that is, the distance prior to $t = 0$.

4. If a contact point of a relay is given an acceleration $\mathbf{a} = 9{,}000t^{1/2}$ centimeters per second, what is its speed after 0.01 second?

5. What distance was traveled by the contact point of Prob. 4 in 0.01 second?

6. The current in a circuit was $i = 4t^3$ amperes. How many coulombs were transmitted in 3 seconds?

7. If a capacitor contains a charge of 0.01 coulomb, and if a current $i = 2t$ amperes is supplied to it, what is the charge after 0.1 second? (Assume the charging current has the same direction as the initial charge.)

8. An 80-microfarad capacitor is charged to a voltage of 100 volts. A current $i = 0.04t^3$ amperes is then supplied to the capacitor in the same direction as the original charge. After what time interval in seconds does the capacitor voltage reach 225 volts?

9. A certain coil has 110 turns. It is situated in a magnetic field such that the flux through the coil is 0.8 weber. If it is now desired to vary the flux so that an emf $v_{ind} = -5t^2$ volts will be induced in the coil, what must be the equation for the flux through the coil?

10. Find the equation for the current i in a 6-henry inductor if there is induced an emf $v_{ind} = t^{5/3} - 5$ volts. (Let $i = 0$ when $t = 0$.)

11. A direct current of 0.3 ampere flows in a 15-henry inductance. Superimposed on this direct current is a varying current such that a voltage $v_{ind} = 120t^{1/3}$ volts is induced. Find the instantaneous total current when $t = 1$ second. (Assume that the steady and the varying components are additive at the instant in question.)

12. The mutual inductance M between the primary and the secondary windings of a transformer is 4 henrys. What primary current i_1 expressed as a function of time is required to produce a secondary emf of the form $t^2 - 3t$ volts? (Assume that no direct current flows in the primary.)

13. The direct current in the plate circuit of an electron tube is 0.05 ampere. The plate circuit includes a transformer which delivers a secondary emf v_2 because of the varying component of plate current. The mutual inductance between the primary and the secondary windings is $\frac{1}{6}$ henry. If it is desired to make v_2 of the form $0.1 - 0.1t + 0.05t^2$ volts, find the equation of the plate current.

14. If the power in a circuit varied from $t = 0$ to $t = 2$ seconds according to $p = 500\ (3t^2 + t)$ watts, find the energy expended.

15. The voltage supplied to a circuit was $v = 2t + 1$ volts. If the cur-

rent was $i = 0.03t$ amperes, find the energy w delivered from $t = 0$ to $t = 50$ seconds.

16. A current $i = 3t^2$ amperes flows in a circuit whose resistance varies according to $r = t^{-1/2}$ ohms. Find the energy supplied to the circuit from $t = 0$ to $t = 4$ seconds.

17. A battery charger supplies a current of 10 amperes to a cell for 1 hour; thereafter the current is $i = 11 - t$ amperes, where t is the time in hours. Find the total time at which the charge supplied is 70.5 ampere-hours. [HINT: Clearly, Formula (21) applies if we take time in hours and charge in ampere-hours.]

18. A charged particle has a mass m. If a force $\mathbf{F} = kt^{1/2}$ (k is a constant) is applied to the particle, what formula expresses the distance s which it moves in any time t, starting from rest?

9-7 Integrating circuits. Figure 9-3 illustrates a circuit in which a constant-current generator (such as a pentode) supplies current to a capacitor. According to (22), the voltage across the capacitor will be

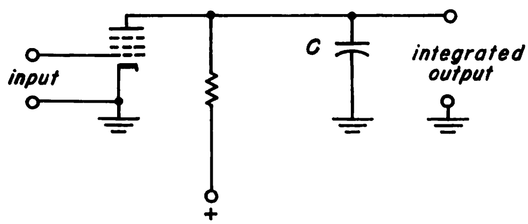


Fig. 9-3

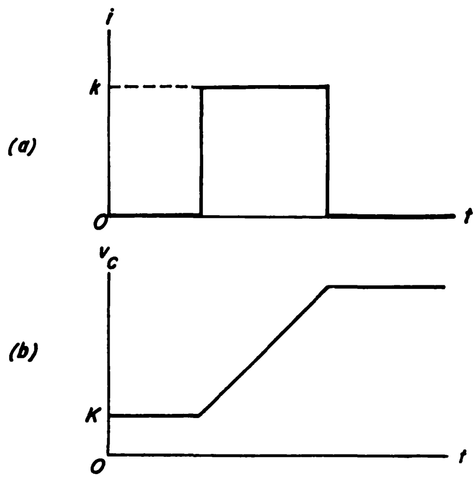


Fig. 9-4

$v_C = (1/C)\int i\,dt$ at any time. Since the voltage is proportional to the integral of the input current, we call such a circuit an *integrating circuit.* Such circuits are widely used, as in television receivers. Suppose, for instance, that we apply to this circuit the rectangular wave of Fig. 9-4a,

having the form

$$i = k = \text{const}$$

We get an output wave like that of Fig. 9-4*b*, having the form

$$v_C = \frac{1}{C}\int k\,dt = \frac{kt}{C} + K$$

The integrating constant K here represents the voltage, if any, to which the capacitor was initially charged before the input signal was applied.

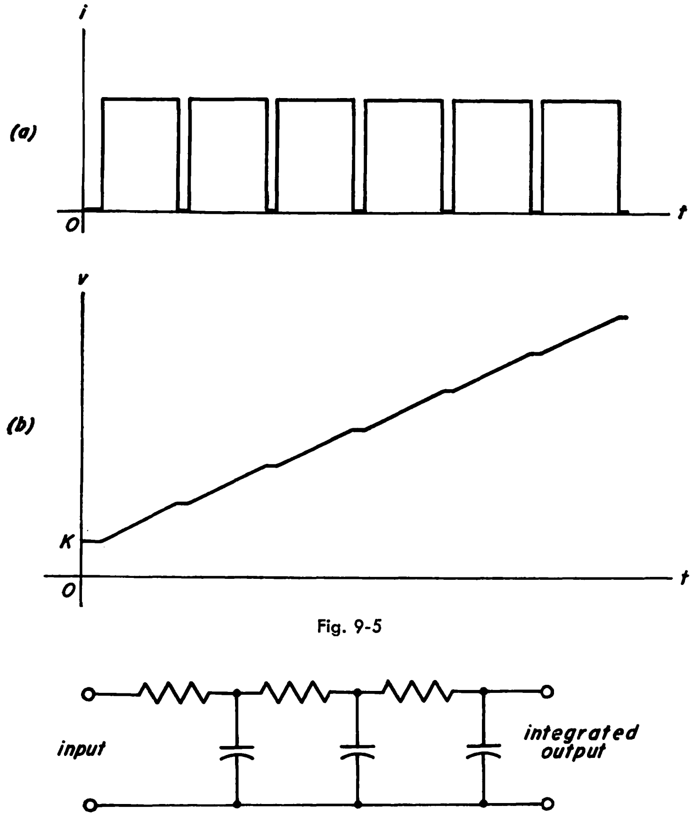


Fig. 9-5

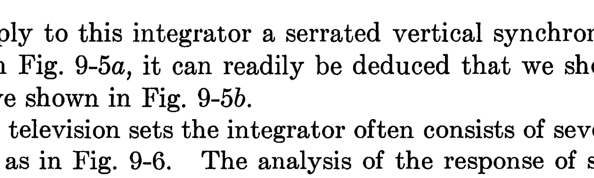


Fig. 9-6

If we apply to this integrator a serrated vertical synchronizing pulse, as shown in Fig. 9-5*a*, it can readily be deduced that we should get the output wave shown in Fig. 9-5*b*.

In actual television sets the integrator often consists of several sections in cascade, as in Fig. 9-6. The analysis of the response of such circuits to various input waveforms is the subject of more advanced study.

It is also presumably possible to use an RL circuit, like that of Fig. 9-7, as an integrator. Let the source shown be a constant-voltage generator (one of negligible internal impedance). Neglecting the resistance of the inductor and the small resistance R, the *applied* voltage needed to produce a given current in the circuit is, by Eq. (37), Sec. 5-14*b*,

$$v_L = L\frac{di}{dt}$$

Rearranging and integrating (and using the symbol i_L to indicate the current through the inductor),

$$i_L = \frac{1}{L}\int v_L\,dt \tag{27}$$

where v_L is the voltage applied to the inductor. By Ohm's law, the output voltage across R will be

$$v_R = Ri_L = \frac{R}{L}\int v_L\,dt \tag{28}$$

The circuit of Fig. 9-7 should then be capable of delivering an output voltage which is proportional to the integral of the input impulse. In a

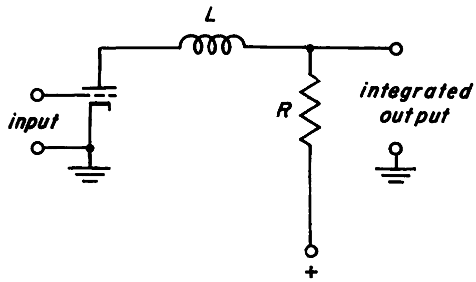


Fig. 9-7

practical circuit, however, the resistance of the inductor is generally far from negligible, so that the accuracy of the integration may suffer. For this reason, and from an economy standpoint, it is usually preferred to use capacitive integrators.

Example 1. The circuit of Fig. 9-8 is useful for displaying the *hysteresis loop* of a sample of magnetic material. The voltage applied to the horizontal oscilloscope plates H, H is proportional to the current supplied and therefore to the magnetic-field intensity $\mathbf{H}$ existing in the magnetic circuit formed by the sample. The voltage induced in the output winding is $v_{ind} = -N\,d\phi/dt$, where N is the number of turns in that winding. If R in the integrating circuit is made large, the current into the integrating capacitor will be nearly proportional to v_{ind}. Then the integrated output signal will be proportional to ϕ itself, rather than to its derivative. It will then be an indication of flux *density* $\mathbf{B}$, so that the shape of the **B-H** curve will be shown on the oscilloscope screen.

Example 2. When a phase modulator is used in a voice radio-communication system to produce equivalent frequency modulation, the resulting carrier-frequency variation Δf at any instant is proportional to the slope dv/dt of the applied modulating voltage wave. It is desired to design a circuit to prevent overmodulation in this system.

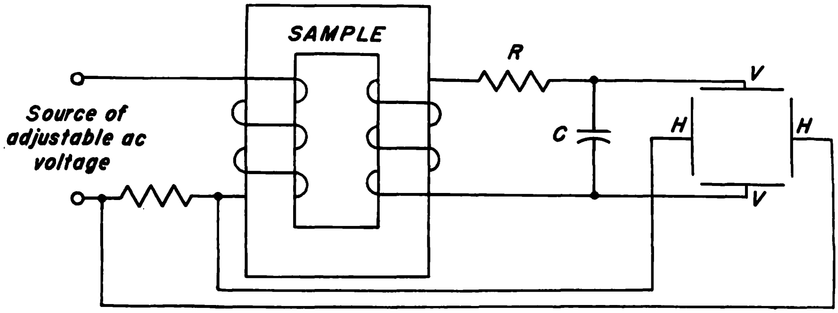


Fig. 9-8

Clearly a simple peak-clipper circuit does not suffice, for it does not prevent the af voltage wave from rising *steeply* to the clipping level. A (patented) circuit, shown in block form in Fig. 9-9, meets the need as follows. The af wave is sent through a differentiating circuit, giving a wave which is always proportional

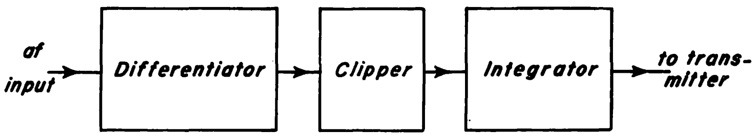


Fig. 9-9

to the *slope* of the original speech wave. To prevent overmodulation of the transmitter, a clipper circuit limits the slope wave to a predetermined amplitude. An integrator then reverses the differentiation process. This gives a wave which is similar to the original speech wave, except that its slope can never become excessive because of the action of the clipper.

PROBLEM

1. Sketch or trace the waveforms of Fig. 9-10, and under each show the wave you would expect to get after integrating the given wave by means of a capacitive integrator.

9-8 Further applications of Kirchhoff's laws. *a. The current law.* Figure 9-11 shows a resistance R connected in parallel with an inductance L. A generator GEN supplies a current i of any physically realizable waveform. Let v represent the voltage of the generator at any instant. The current in the resistor is $i_R = v/R$, while by (27) the current in the

inductance is $i_L = (1/L)\int v\,dt$. Then, according to Kirchhoff's current law,

$$\frac{v}{R} + \frac{1}{L}\int v\,dt - i = 0 \tag{29}$$

Example 1. In the circuit of Fig. 9-11, assume $R = 20$ ohms and $L = 10$ henrys. If a voltage $v = t^3 + 25t$ is supplied by the generator, and if $i = 0$ when $t = 0$, find i when $t = 2$ seconds.

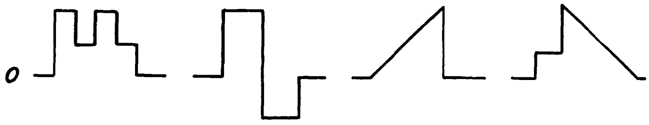


Fig. 9-10

We find $\int v\,dt = \int (t^3 + 25t)\,dt = t^4/4 + 25t^2/2 + K$. By (29), this gives

$$\frac{1}{20}(t^2 + 25t) + \frac{1}{10}\left(\frac{t^4}{4} + 12.5t^2 + K\right) - i = 0$$

For convenience, let $K/10$ be represented by K_1. Rewriting, we get

$$i = 0.025t^4 + 0.05t^3 + 1.25t^2 + 1.25t + K_1$$

Since $i = 0$ when $t = 0$, we get $K_1 = 0$. Letting $t = 2$, we have $i = 8.3$ amperes.

If to the circuit of Fig. 9-11 we should connect a shunt capacitance C,

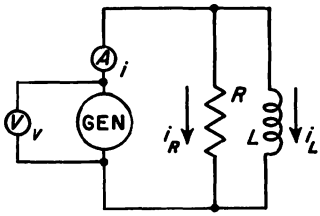


Fig. 9-11

this capacitance will take a current $i_C = C\,dv/dt$.

$$C\frac{dv}{dt} + \frac{v}{R} + \frac{1}{L}\int v\,dt - i = 0 \tag{30}$$

Example 2. Suppose that a capacitor $C = 20$ microfarads, a resistor $R = 10{,}000$ ohms, and an inductor $L = 100$ henrys are connected in parallel. If the applied emf is $v = 60t^2$, and if $i = 0$ when $t = 0$, find i when $t = 1$ second.

Since $v = 60t^2$, we find $dv/dt = 120t$, and $\int v\,dt = 20t^3 + K$. By (30),

$$2 \times 10^{-5}(120t) + 10^{-4}(60t^2) + 10^{-2}(20t^3 + K) - i = 0$$

Let $K_1 = 10^{-2}K$. Solving the above equation for i,

$$i = 2 \times 10^{-1}t^3 + 6 \times 10^{-3}t^2 + 2.4 \times 10^{-3}t + K_1$$

Since $i = 0$ when $t = 0$, we get $K_1 = 0$. Setting $t = 1$, we find $i = 0.2084$ ampere.

b. The voltage law. If a current i is supplied at a voltage v from a generator GEN to a series RC circuit, as in Fig. 9-12, and if the sum of the

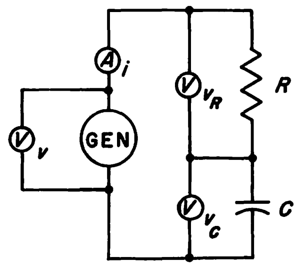


Fig. 9-12

resulting voltage drops around the circuit is equated to zero, we get

$$Ri + \frac{1}{C}\int i\,dt - v = 0 \tag{31}$$

If an inductance L is connected in series with this circuit, the following equation applies:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt - v = 0 \tag{32}$$

QUESTIONS

1. What equation relates the currents in the branches of a parallel RL circuit? Of a parallel RLC circuit?

2. State an equation relating the voltage drops around a series RC circuit; a series RLC circuit.

3. Note that Eq. (31) of Sec. 9-8 describes the operation of the differentiating circuit of Fig. 5-11 in more detail than does Eq. (39) of Sec. 5-15. Solve Eq. (31) of Sec. 9-8 for the derivative of the output voltage Ri of this circuit, and point out that this output voltage cannot change more rapidly than does the output voltage v of the tube.

PROBLEMS

1. An inductance of 8 henrys is connected in parallel with a resistance of 12 ohms. A voltage $v = 20t^2$ is applied to this combination. If $i = 0$ when $t = 0$, find a formula for i.

2. If a voltage $v = 90\sqrt{t}$ is applied to a circuit consisting of a 30-henry inductance shunted by a 50-ohm resistance, what value of current flows when $t = 4$ seconds? Let $i = 0$ when $t = 0$.

3. If a voltage $v = 20t^4$ is applied across a parallel RL combination, where $R = 500$ ohms and $L = 40$ henrys, find the total current i when $t = 0.2$ second. Let $i = 4$ microamperes when $t = 0$.

4. In the circuit of Fig. 9-11, let $R = 5$ ohms and $L = 0.2$ henry. If $v = \sqrt{t^3} + 2$ volts, find i when $t = 4$ seconds. Assume $i = 0.4$ ampere when $t = 0$.

5. A current $i = 0.005t^{1/2}$ amperes flows in the circuit of Fig. 9-12, where $R = 8.8 \times 10^4$ ohms and $C = 1$ microfarad. Find a formula for the impressed voltage as a function of time t. Assume the capacitor to be initially discharged.

6. In Fig. 9-12 let $R = 2 \times 10^5$ ohms and $C = 2$ microfarads. If the current in the circuit is $10^{-4}t^{1/3}$ amperes, and if the capacitor is initially charged to 225 volts, find the voltage across the circuit terminals when $t = 8$ seconds.

7. Find the total current in a parallel RLC circuit when $t = 0.5$ second if a voltage $v = 400t^3$ volts is impressed across the circuit. Let $C = 80$ microfarads, $R = 2 \times 10^4$ ohms, and $L = 125$ henrys, and let $i = 0$ when $t = 0$.

8. A series circuit has these constants: $R = 5 \times 10^3$ ohms, $L = 200$ henrys, and $C = 20$ microfarads. If a current $i = 0.02t^2$ amperes is supplied to the circuit, at what rate is the voltage across the circuit changing when $t = 0.1$ second?

9-9 Conclusion. In mathematics certain operations are considered to be the reverse (or inverse) of other operations. If, for instance, we differentiate a certain function, we get a second function, called the derivative of the first. If we then take this derivative, as an integrand, and perform an operation which is the reverse of differentiation, that is, integration, upon it, we get the original function (to within an arbitrary constant). Integration, which is the more difficult of the two operations, is usually spoken of as the inverse operation. The integral obtained from a given integrand is often called the *antiderivative* of the integrand, or its *primitive function.*

We may look upon this inverse relationship much as we think of division as being the inverse of multiplication or as we consider the extracting of a square root as the inverse of the squaring of a quantity.

Integrals are often expressed in functional notation. The integral $\int f(x)\,dx$ is often represented by $F(x) + C$. Another notation used for $\int f(x)\,dx$ is $f^{-1}(x)$. In the latter notation, the value taken by the integral when x is equal to, say 3, would be expressed as $f^{-1}(3)$.

It may be shown that a function is integrable (possesses an integral) if it is continuous, that is, if its graph shows no *breaks* or abrupt *jumps.*

Note that a basic idea in integration is the *guessing* of a function which will have a given derivative. In practice, the operation of integration generally reduces to that of transforming the function to be integrated into some *type form,* which may be integrated according to standard rules. In addition to the integration rules developed in the present chapter, we shall later devise rules for the integration of further functions. (In

particular, note that we give no general rules for integrating the product or the quotient of two functions.)

If we should integrate a function by two different methods, we might obtain two apparently different results. Despite their varying appearances, however, these results would be equal to within an arbitrary constant. A very simple example of this situation follows.

Example. Find $\int (x^2 + 5)x\,dx$.

We shall carry out this integration by two different procedures. First, we multiply out the integrand, getting

$$\int (x^3 + 5x)\,dx = \frac{x^4}{4} + \frac{5x^2}{2} + C$$

As an alternative procedure, we may first note that if we let $u = x^2 + 5$ in the given integral, then $du = 2x\,dx$. Multiplying and dividing by 2,

$$\frac{1}{2}\int (x^2 + 5)2x\,dx = \frac{1}{4}(x^2 + 5)^2 + C_1 = \frac{x^4}{4} + \frac{5x^2}{2} + 25 + C_1$$

The results obtained by the two different procedures differ only by the constant difference 25. (In a practical problem, this difference can be adjusted by assigning a suitable value to the integration constant.)

10

Definite Integrals

Next we shall observe some facts about integration which will help to systematize the solution of many problems.

10-1 The definite integral. Let it be given that

$$dy = 3x^2\,dx \tag{1}$$

and that $$y = 0 \qquad \text{when } x = 3 \tag{2}$$

Let it be desired to find the value of y when $x = 5$.

Integrating both sides of (1),

$$y = x^3 + C \tag{3}$$

Substituting the initial values (2),

$$0 = 3^3 + C \qquad \text{or} \qquad C = -27$$

This gives

$$y = x^3 - 27 \tag{4}$$

Thus, when $x = 5$,

$$y = 5^3 - 27 = 98 \tag{5}$$

Observe that the result (5) can also be obtained by the following procedure:

First, integrating in (1), *neglecting* the integration constant C, getting

$$y = x^3 \tag{6}$$

Second, finding the *difference* between the value of (6) when $x = 5$ and that obtained when $x = 3$:

$$y \text{ (when } x = 5) = 5^3 - 3^3 = 98 \tag{7}$$

This difference is called the *definite integral from* 3 *to* 5 *of* $3x^2\,dx$, and is written

$$\int_3^5 3x^2\,dx = 98 \tag{8}$$

The quantities 3 and 5 associated with the integral sign are called the *end values* or the *lower* and *upper limits,* respectively, of the integral. (This use of *limit* is not connected with the idea of a limit approached by a variable.)

In working problems like the above, it is convenient to use symbols like

$$x^3 \Big]_3^5 \tag{9}$$

to indicate "the value of x^3 when $x = 5$, minus the value of x^3 when $x = 3$." The solution of the above problem would then be written

$$\int_3^5 3x^2\,dx = x^3 \Big]_3^5 = 5^3 - 3^3 = 98 \tag{10}$$

(In more complicated expressions, we avoid confusion by *indicating* the quantity to which the limits are applied. To indicate, for instance, the difference between the values of ux^2 when $x = 4$ and when $x = 2$, we could write $ux^2\,]_{x=2}^{4}$.)

Note that

➤ No constant of integration is written when the definite-integral form is used.

This constant is taken care of when we substitute the lower limit in the integral.

Example. If the power in a circuit varied according to $p = t^2 - t$ watts, find the total energy w expended from $t = 1$ to $t = 4$ seconds.

Note that the energy is taken into account only after $t = 1$; that is, $w = 0$ when $t = 1$. The lower limit of integration is therefore $t = 1$. We write

$$w = \int_1^4 (t^2 - t)\,dt = \left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Big]_1^4 = (64/3 - 8) - (1/3 - 1/2) = 27/2 \text{ joules}$$

We see, then, that solutions of problems of the above kind can be written in either the *indefinite-integral* form shown in Chap. 9, or, which

is often more convenient, in the *definite-integral* form just given.

The value of a definite integral is a quantity which depends upon the limits of integration but not upon the variable in the integrand. For instance, we get the same result from both the integrals

$$\int_0^1 x^2\,dx \qquad \text{and} \qquad \int_0^1 t^2\,dt$$

as you can check.*

If the upper and lower limits are interchanged, it will be found that the value of the definite integral is changed only in *sign.*

QUESTIONS

1. In a certain problem it was desired to evaluate the difference between the value of the integral of a given function of x when $x = m$ and the value of that integral when $x = n$ (omitting, in both cases, any constant of integration). What is this difference called?

2. In what way, if any, are the upper and lower limits of a definite integral related to the limits approached by variables in certain cases?

3. In what kind of integral is the constant of integration omitted?

4. Does the value of a definite integral depend upon which variable appears in the integrand? If not, upon what does the value of the integral depend?

5. If the upper and lower limits of a definite integral are interchanged, what is the effect upon the value of the integral?

PROBLEMS

Evaluate the following definite integrals. Write the solutions in the form given in Eq. (10).

1. $\int_0^2 x\,dx$
2. $\int_1^{10} dx$
3. $\int_0^5 t^3\,dt$
4. $\int_2^3 x^5\,dx$
5. $\int_{-2}^2 y^2\,dy$
6. $\int_0^{100} dz$
7. $\int_1^4 x^3\,dx$
8. $\int_{-1}^3 (1 + 3t^2)\,dt$
9. $\int_2^6 (6u^2 - 2u)\,du$
10. $\int_1^7 (4v^3 + 5)\,dv$
11. $\int_0^\pi (8\theta^3 + 4\theta)\,d\theta$
12. $\int_0^2 (y^5 - 2)\,dy$

10-2 Average height of a curve. We shall often have occasion to refer to the *average height* of a graph over a given interval. Just what does this term mean?

* References 6, 7, 9, and 10 of Chap. 1 include many definite integrals. Reference 1 of the present chapter is a large standard table of definite integrals. This reference utilizes certain symbols which are archaic. In addition, the text material is principally in French. These comments apply also to ref. 2, which is intended to supplement and to correct certain parts of ref. 1. Nevertheless, refs. 1 and 2 are widely used by advanced workers. Certain definite integrals called *improper integrals* are tabulated in recent books cited as ref. 5 of Chap. 15.

Figure 10-1 shows the graph of a function $y = f(x)$. Consider the area A described by the graph and the x axis from the point where $x = a$ to the point where $x = b$. (Here we have made $a = 2$ and $b = 6$.)

➢ We define the average height of the graph, over the range from a to b, as the area A divided by the length of the base $b - a$.

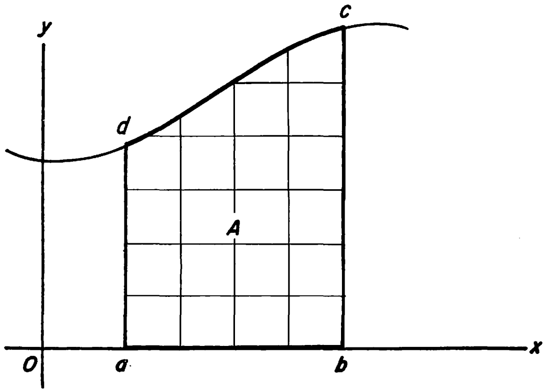


Fig. 10-1

For example, let each square in the figure represent 1 unit of area. If the figure $abcd$ contains an area of 19.8 units, then the average height of the curve from $x = 2$ to $x = 6$ is taken as $19.8/(6 - 2) = 4.95$ units.

10-3 Area under a curve. Consider a function $y = f(x)$, as graphed in Fig. 10-2. Let us direct our attention to the area A beneath this graph, and let us study the manner in which A changes with x.

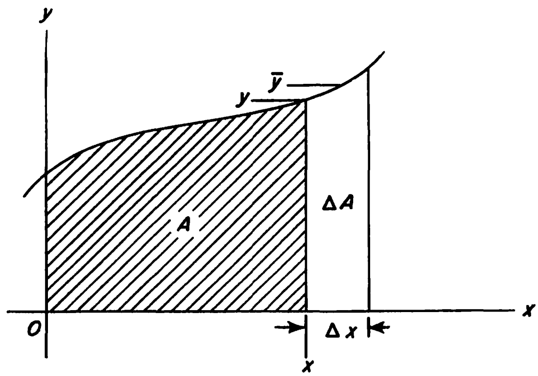


Fig. 10-2

When the independent variable has the value x shown in the figure, the height of the curve is indicated by y, and the area A is that indicated by the shaded region. If we now let x increase by an amount Δx, we get an increase ΔA in the area.

Let the average value of y over the range Δx be indicated by $\bar{y}$ (read "y bar"). Then by the definition in Sec. 10-2,

$$\frac{\Delta A}{\Delta x} = \bar{y} \tag{11}$$

If we now let Δx approach zero, the left member of (11) approaches dA/dx. And the right member $\bar{y}$ approaches y. Thus

$$\frac{dA}{dx} = y \tag{12}$$

That is,

The rate of change of the area under a curve is equal at any point to the height of the curve at that point.

This also gives the important result that

$$A = \int y\,dx \tag{13}$$

That is,

The integral of a function is given by the area under its graph.

Example 1. Find the area A under the graph of the function $y = x^2$ from $x = 1$ to $x = 2$.

This graph is shown in Fig. 10-3. We find the desired area by (13), using the definite-integral notation. We note that the upper limit of integration is 2 and that, since $A = 0$ when $x = 1$, the lower limit is 1:

$$A = \int_1^2 x^2\,dx = \frac{x^3}{3}\Bigg]_1^2 = 7/3 \text{ area units}$$

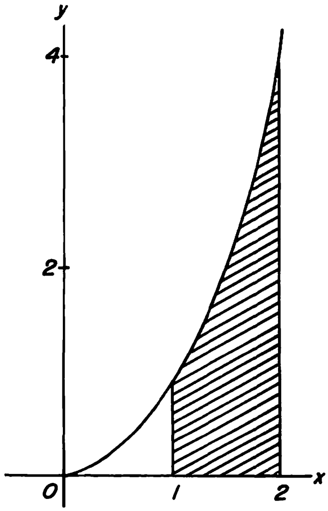


Fig. 10-3

Example 2. The power in a circuit varied according to $p = t^2$. Find the average power P_{av} from $t = 2$ to $t = 4$ seconds.

The function $p = t^2$ is graphed in Fig. 10-4. Clearly, the average power is given by the average height of the graph over the given interval. We first find the *area* w under the graph by (13):

$$w = \int_2^4 t^2\,dt = \frac{t^3}{3}\Bigg]_2^4 = 56/3 \text{ units}$$

[By Eq. (26), Sec. 9-6, this area gives the energy w in joules expended during the given interval.] If we divide w by the length of the interval, we get by Sec. 10-2 the average power:

$$P_{av} = \frac{1}{4-2}\int_2^4 t^2\,dt = 28/3 \text{ watts}$$

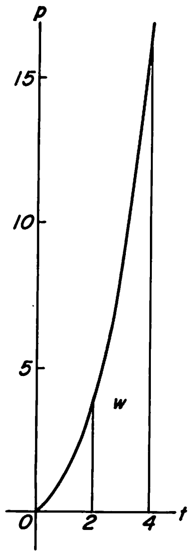


Fig. 10-4

Similarly, the average power in *any* circuit over a time interval from $t = a$ to $t = b$ is

$$P_{\text{av}} = \frac{1}{b - a} \int_a^b p\, dt \qquad \text{watts} \tag{14}$$

It is, by the way, this *average* power which indicates the energy-delivering effect in the circuit, expressions like "effective power" being seldom used.

If we want to evaluate an average voltage or current, we can use formulas similar in form to (14):

$$I_{\text{av}} = \frac{1}{b - a} \int_a^b i\, dt \tag{15}$$

$$V_{\text{av}} = \frac{1}{b - a} \int_a^b v\, dt \tag{16}$$

10-4 Area under a curve versus area between the curve and the x axis. Consider the function $y = 1 - x^2$, graphed in Fig. 10-5. Let it be desired to find the area A_1 under this curve from $x = 0$ to $x = +1$. In accordance with Sec. 10-3, we can obtain this area by

$$A_1 = \int_0^1 (1 - x^2)\, dx = \left(x - \frac{x^3}{3}\right)\Big]_0^1 = \tfrac{2}{3} \text{ area unit}$$

Next, let it be desired to find the area under this graph from $x = +1$ to $x = +3$. Proceeding as before, and calling this new area A_2,

$$A_2 = \int_1^3 (1 - x^2)\, dx = \left(x - \frac{x^3}{3}\right)\Big]_1^3 = -2\tfrac{2}{3} \text{ area units}$$

The *minus* sign preceding this result is associated with the fact that the area A_2 lies *above* the curve (and *below* the x axis), being, so to speak, a *negative* value of area *beneath* the curve.

If now we should evaluate the definite integral $\int_0^3 (1 - x^2)\, dx$, we should expect a result which expresses an area A equal to the *algebraic* sum of A_1 and A_2, and this is just the result we get:

$$A = \int_0^3 (1 - x^2)\, dx = \left(x - \frac{x^3}{3}\right)\Big]_0^3 = -6 \text{ area units}$$

In general, if $y = f(x)$, then the definite integral from a to b of y

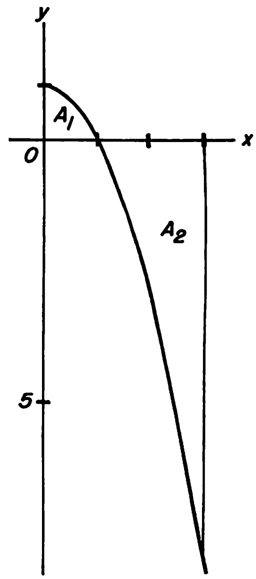


Fig. 10-5

with respect to x gives the area under the graph of y from $x = a$ to $x = b$, with areas which appear *above* the graph being included as *negative* areas.

Example 1. The current delivered to a capacitor varied as $i = 10^{-2}(t^3 - 3t^2 + 2t)$. If the charge q in the circuit was zero when $t = 0$, find q when $t = 3$ seconds.

According to the given formula, i is actually *negative* during a portion of the interval from $t = 0$ to $t = 3$ seconds, indicating a direction of flow such that the capacitor is discharging during that part of the interval. However, to obtain the over-all charge q which flows *into* the capacitor, we simply evaluate

$$q = 10^{-2} \int_0^3 (t^3 - 3t^2 + 2t)\,dt = 10^{-2}\left(\frac{t^4}{4} - t^3 + t^2\right)\Bigg]_0^3 = 0.0225 \text{ coulomb}$$

Returning to Fig. 10-5, let us consider a different problem. Suppose that we wanted to find the sum A_T of A_1 and A_2 *taking both areas as positive.* Clearly this result can be had by adding the numerical values* of A_1 and A_2:

$$\begin{aligned} A_T &= |A_1| + |A_2| = \left| \int_0^1 (1 - x^2)\,dx \right| + \left| \int_1^3 (1 - x^2)\,dx \right| \\ &= 2/3 + 20/3 = 22/3 \text{ area units} \end{aligned}$$

To get the total area from $x = a$ to $x = b$ between a graph of $y = f(x)$ and the x axis (taking areas both above and below the graph as positive) we must (a) find any values of x in this interval for which the curve crosses the x axis, (b) evaluate $\int y\,dx$ over the separate subintervals thus obtained, and (c) add the numerical values of these integrals.

Example 2. Find the sum q_T of the numerical values of the charges which flowed into and out of the capacitor of Example 1 from $t = 0$ to $t = 3$.

We find the points at which the graph of i encounters the t axis by setting $i = 0$:

$$\begin{gathered} i = 10^{-2}(t^3 - 3t^2 + 2t) = 0 \\ t(t-1)(t-2) = 0 \\ t = 0, 1, 2 \end{gathered}$$

Thus,

$$\begin{aligned} q_T &= 10^{-2}\left[\left| \int_0^1 (t^3 - 3t^2 + 2t)\,dt \right| + \left| \int_1^2 (t^3 - 3t^2 + 2t)\,dt \right| \right. \\ &\qquad \left. + \left| \int_2^3 (t^3 - 3t^2 + 2t)\,dt \right| \right] \\ &= 10^{-2}(1/4 + 1/4 + 9/4) = 0.0275 \text{ coulomb} \end{aligned}$$

QUESTIONS

1. Define the *average height* of a curve.
2. What expression gives the rate of change of the area under a curve?

* The symbol $|N|$ indicates the *numerical* or *absolute* value of a quantity N, regardless of its sign.

3. What is the relation between the area under a curve and the integral of the function of which the curve is the graph?

4. If we take the definite integral of a function $f(x)$ between the limits $x = a$ and $x = b$, in what way does the result include any areas beneath the x axis and above the graph?

5. State the procedure, using definite integrals, for obtaining the area *between* the graph of a function and the x axis, taking areas both above and below the axis as positive.

PROBLEMS

Solve the following problems, using the definite-integral notation.

1. Find the area under the curve $y = x^{1/2}$ from $x = 4$ to $x = 9$.

2. Find the total area between the curve $y = 3x^2 - 12x + 9$ and the x axis from $x = 0$ to $x = 4$. Treat all areas here as positive.

3. An object fell at a speed $v = 32t$ feet per second. What distance did it traverse from $t = 1$ to $t = 4$ seconds?

4. A plunger was operated by a solenoid so that its acceleration was $10t$ centimeters per second per second. By what amount did its speed increase from $t = 0.5$ to $t = 1$ second?

5. A circuit carried a current $i = 6t^{3/2}$ amperes. Find the charge in coulombs transferred in the interval from $t = 4$ to $t = 9$ seconds.

6. The current in a circuit was $i = 3t^2 - 10t + 7$ amperes. (*a*) What amount of charge was transferred from $t = 1$ to $t = 4$ seconds, taking into account the direction of flow? (*b*) What was the average current over this interval?

7. In the circuit of Prob. 6 find the total charge transfered during the interval, disregarding the direction of flow.

8. Fuel was pumped into a storage tank at a rate of flow $F = 3t^2 - 15t + 18$ gallons per minute. (Note that during a part of the time F was negative, indicating that fuel was being pumped out of the tank.) (*a*) Find the amount of fuel added to the tank from $t = 1$ to $t = 5$ minutes. (*b*) Find the average rate of flow over this interval.

9. Find the total number of gallons of fuel pumped in the interval discussed in Prob. 8, regardless of the direction of flow.

10. The power in a circuit was $p = 10t^3$ watts. (*a*) Find the energy used from $t = 3$ to $t = 5$ seconds. (*b*) Find the average power during this interval.

11. By what amount did the current in a 2-henry inductor change from $t = 1$ to $t = 6$ seconds if the induced emf during this interval was $v_{ind} = 2 - t^2$ volts?

12. Find the charge q in coulombs transmitted from $t = 1$ to $t = 6$ seconds through the inductor in Prob. 11, taking into account the direction of flow. Let $i = 0$ and $v_{ind} = 0$ when $t = 0$.

10-5 The definite integral as the limit of a sum; the fundamental theorem. Figure 10-6*a* presents the graph of a function $y = f(x)$. Here, the region from $x = a$ to $x = b$ along the x axis is divided into n equal small intervals Δx. The height of the curve at the beginning of each such interval is y_1, y_2, or y_3, etc. Considering the area A under the curve from $x = a$ to $x = b$, we find A divided into small portions each approximately rectangular in shape, and each having base Δx and height y_1, y_2, or y_3, etc. In fact, if we take the sum of the areas of the unshaded *actual* rectangles shown, this sum will approximate the area beneath the curve. That is,

$$A \approx y_1\,\Delta x + y_2\,\Delta x + y_3\,\Delta x + \cdots + y_n\,\Delta x \tag{17}$$

The amount of error in (17) is indicated by the sum of the small shaded sections.

Now, in diagram (*b*) we have subdivided each of the intervals of diagram (*a*), so that Δx is smaller than before. Again, the sum of the actual rectangles in diagram (*b*) is approximately the area under the curve—

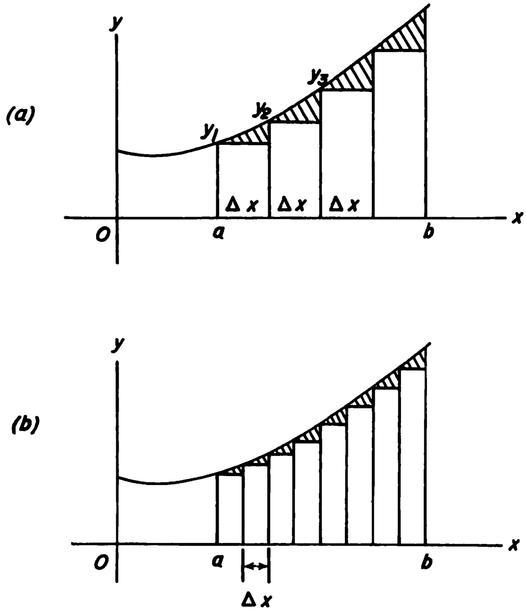


Fig. 10-6

expressed as in (17), and again the error in this approximation is the total area of the small shaded sections. We see that the error in diagram (*b*) is smaller than that in diagram (*a*), and we are led to believe that if we should use a greater and greater number n of smaller and smaller rectangles, letting Δx approach zero, then the error would approach zero. That is,

The area A under the curve must equal the limit approached by the sum of all the rectangles:

$$A = \lim_{\Delta x \to 0} (y_1\,\Delta x + y_2\,\Delta x + \cdots + y_n\,\Delta x) \tag{18}$$

The result (18) leads us to what is called *the fundamental theorem of integral calculus.** For, by (13), the area A expresses the definite integral

* The presentation here given for this theorem lacks generality, but it serves to illustrate the result. An adequate proof of the theorem is found in more formal texts.

$\int_a^b y\,dx$, so that

$$\lim_{\Delta x \to 0} (y_1\,\Delta x + y_2\,\Delta x + \cdots + y_n\,\Delta x) = \int_a^b y\,dx \tag{19}$$

The statement of (19), namely, that any quantity expressible as the limit of a sum of the kind shown is equal to the definite integral $\int_a^b y\,dx$, is of basic importance. For one thing, we find in it a second definition of the meaning of an integral:

The integral from a to b of $y\,dx$ is equal to the limit approached by the sum of the products of the form $y\,\Delta x$ over the same interval, as $\Delta x \to 0$.

In Chap. 9, an integral was defined as a quantity having a given differential. Here we have the integral as the limit approached by a certain kind of sum.

The above theorem permits us easily to set up many integrals. For instance, the formula

$$q = \int_{t=a}^{b} i\,dt$$

could be derived by considering the region from $t = a$ to $t = b$ as being divided into short intervals Δt. The charge transferred during each interval is then equal approximately to the current (say i_1) at the beginning of the interval, multiplied by Δt. Then the total charge transmitted from $t = a$ to $t = b$ must be approximately $i_1\,\Delta t + i_2\,\Delta t + \cdots + i_n\,\Delta t$. The error in this approximation becomes less as Δt is taken smaller. We take the total charge, then, as

$$q = \lim_{\Delta t \to 0} (i_1\,\Delta t + i_2\,\Delta t + \cdots + i_n\,\Delta t) = \int_a^b i\,dt$$

A shorter, *inexact* method of reasoning which enables us to set up such integrals still more quickly is described in the next section.

10-6 Infinitesimal elements. *a. Area under a curve.* Integral formulas to fit physical problems, like the area problem of Fig. 10-6 above, are often derived by a shorter (although somewhat improper) method of reasoning.

In Fig. 10-6 consider the range from $x = a$ to $x = b$ to be divided into *very small* intervals, and call the length of each interval dx. Think of these intervals as being so small that the height y of the curve *does not vary** over the interval dx. Then the area dA of one of the little rectangular strips, or so-called *infinitesimal elements* of area, is $y\,dx$.

* We know this to be impossible, but this rather faulty thinking leads us quickly to correct results. It is permissible if we understand that the significance of the process is really that presented in *limit* terms in Sec. 10-5. The method of Sec. 10-6 is customary among engineers and scientists because of the quickness with which it gets results.

For present purposes, let $\int_a^b y\,dx$ represent the *sum* of all the little intervals dx, each multiplied by the height y during the interval. (The $\int$ sign was originally an elongated S, for the Latin *summa*, sum.) Considering the total area to be represented by the sum of the elementary strips, we get

$$A = \int_a^b y\,dx \tag{20}$$

The above crude reasoning was actually used by some of the early workers with calculus. Today, it is only a short cut. Likewise, differentials like dx are *not* now necessarily taken as being small, except in the inaccurate but brief treatments like that used here to get Formula (20). When we use this shorter method, we must always remember that the actual meaning is that which led us by *limit* considerations to Formula (19).

b. Work done by a varying force. Example 1 below shows how a formula is found for the work done by a varying force.

Example 1. The work done when a constant force $\mathbf{F}$ moves an object over a distance s is

$$w = \mathbf{F}s \tag{21}$$

But in many cases $\mathbf{F}$ is not constant; it may vary as a function of s. Find a formula for w in such a case.

Let the range in s (say from $s = a$ to $s = b$) over which the force operates be broken up into tiny intervals of length ds, so small that $\mathbf{F}$ "does not vary" over one interval. The work performed over one interval is, by (21), $dw = \mathbf{F}\,ds$. The "sum" of all these tiny amounts of work is the total work done:

$$w = \int_a^b \mathbf{F}\,ds \tag{22}$$

c. Volume. We next use the same approximate method of reasoning to get a formula for the volume of a solid. In Fig. 10-7, for instance, the

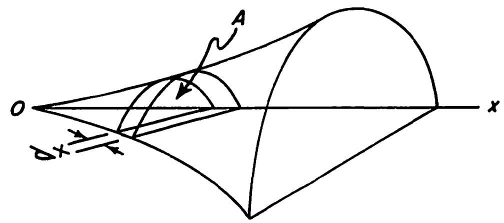


Fig. 10-7

solid shown can be thought of as being made up of very many tiny slices, each having a thickness dx so small that the cross-sectional area A of the slice is "constant" over the interval dx. Then the volume of a slice is $dV = A\,dx$, and the sum of all these infinitesimal elements of

volume is the total volume:

$$V = \int_a^b A\,dx \qquad (23)$$

The use of this important formula is illustrated in the following example.

Example 2. The radius of a paraboloidal reflector varied with distance x from its apex O according to $r = 4\sqrt{x}$. Find the volume contained by the reflector if its depth was 5 units.

The area of the circular cross section of the reflector at any point was, of course, $A = \pi r^2$. Substituting $r = 4\sqrt{x}$, we find $A = 16\pi x$. The volume, by (23), was

$$V = \int_0^5 A\,dx = 16\pi \int_0^5 x\,dx = 8\pi x^2\Big]_0^5 = 200\pi \text{ volume units}$$

QUESTIONS

1. If a certain quantity can be shown to be equal to

$$\lim_{\Delta x \to 0} (y_1\,\Delta x + y_2\,\Delta x + \cdots + y_n\,\Delta x)$$

to what integral must this quantity therefore be equal (fundamental theorem of integral calculus)?

2. Give the brief or approximate reasoning, using infinitesimal elements, showing that the area under a curve is equal to a definite integral.

PROBLEMS

1. The force with which a piston was driven was $\mathbf{F} = 5{,}000s^{-2/3}$ pounds, where s was in inches. Find the work in foot-pounds done over the interval from $s = 1$ to $s = 27$.

2. In stretching a certain spring, the required force $\mathbf{F}$ was proportional to the distance s by which the spring was stretched. If $\mathbf{F} = 25$ pounds when $s = 1$ inch, what work in foot-pounds was required to stretch the spring by 6 inches, starting from normal?

3. When a particle carrying a charge q coulombs is situated in an electric field of intensity $\mathbf{E}$ volts per meter, a force $\mathbf{F} = \mathbf{E}q$ newtons acts upon the particle. Find the work in joules done when an alpha particle ($q = 3.204 \times 10^{-19}$ coulomb) is moved through a distance from $s = 1$ to $s = 3$ meters in an electric field whose intensity varies as $\mathbf{E} = 2{,}000s$ volts per meter.

4. Show that the volume of a cone used in a broadband antenna is $\pi R^2 h/3$, where R is the radius of the base and h is the height. [HINT: By similar triangles, the radius of a circular cross section at any distance x from the apex is $r = kx$, where k is a constant of proportionality. When $x = h$, $r = R$, so that $k = R/h$. Apply (23).]

5. A loudspeaker horn has a radius at any distance s along its axis given by $r = 0.1s^2$ inches. Find the volume of the horn if it extends from $s = 1$ to $s = 10$ inches.

6. A certain pair of charged particles separated by a distance s meters attract each other with a force $\mathbf{F} = 5.25 \times 10^{-21}s^{-2}$ newton. Find the work in joules done in separating them over an interval from $s = 0.01$ to $s = 0.1$ meter.

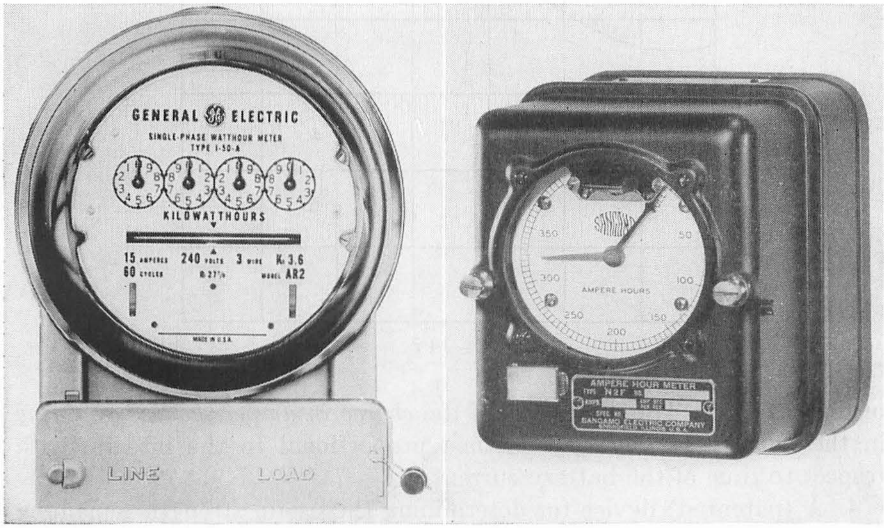
7. If a constant force $\mathbf{F}$ acts for a time t upon a moving object, the momentum given the object is $M = \mathbf{F}t$. (*a*) By the approximate reasoning used in the preceding section, find a formula for M when $\mathbf{F}$ varies as a function of t from $t = 0$ to $t = t_1$. (*b*) Using the formula just developed, find the momentum imparted by a solenoid which applies a force $\mathbf{F} = t^2/2$ pounds to a plunger from $t = 0$ to $t = 0.1$ second.

8. Find a formula for the force $\mathbf{F}$ produced by water pressure against a hydroelectric dam whose length s varies as a known function of a depth h. (HINT: Consider a very narrow strip of width dh and length s across the dam at a depth h. Since water weighs about 62.5 pounds per cubic foot, the pressure at a depth h is $62.5h$ pounds per square foot. The force against the strip is the product of pressure times the area $s\,dh$ of the strip. "Sum up" the forces acting against all such strips from $h = 0$ to the desired depth, $h = h_1$.)

9. Using the formula found in Prob. 8, find the force against the dam if its length varies as $s = 550 - 3h$ feet down to a depth of 35 feet.

10-7 Integrating instruments.

Many instruments in common use yield indications in terms of the *integrals* of the functions fed into them. A few of these will be mentioned.



(a) *General Electric Co.* (b) *Sangamo Electric Co.*

Fig. 10-8

1. The pointer in an automobile speedometer indicates the *speed* of the vehicle at any time; but an additional mechanism, an *odometer*, is included in the instrument, and it gives a reading of *distance* traveled. This indication is visible through a slotted window in the face of the instrument. The distance indication is, of course, the integral of the speed and can be thought of as the sum of all the small intervals of time, each multiplied by the speed of the vehicle over the interval.

2. Figure 10-8*a* shows a kilowatt-hour meter, or *integrating wattmeter*. This familiar device includes a motor mechanism which turns the point-

ers at a rate determined by the power in the circuit. The resulting indication of total energy can be considered as the sum of all the brief time intervals during which the circuit is in operation, each interval being multiplied by the amount of power during the interval. The reading, then, is the integral of the function which would be indicated by an ordinary wattmeter in the same circuit.

3. In using a storage battery, an *ampere-hour meter* (Fig. 10-8*b*) is useful in indicating the condition of charge of the battery. A motor mechanism in the instrument turns the pointer at a rate determined by the current, the direction of rotation depending upon whether the battery is charging

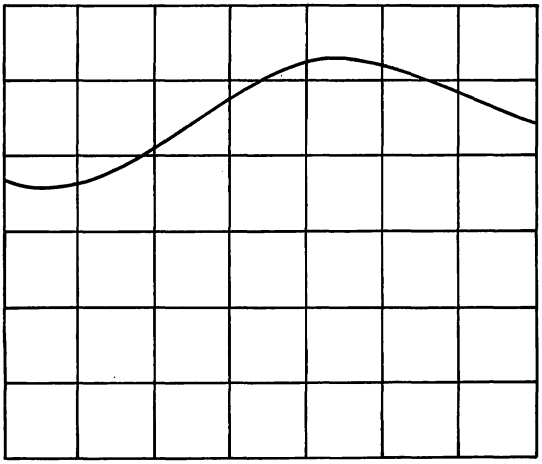


Fig. 10-9

or discharging. The indication of the *charge in ampere-hours remaining* in the battery at any time is then proportional to the integral with respect to time of the battery current.

4. A (patented) device for determining the useful strength remaining in a communication pole makes use of the principle of integration. A small hole is drilled through the pole, and the *strength* indication is the sum of all the tiny distances traveled by the drill, each automatically multiplied by the work required to operate the drill through that distance.

10-8 Graphical integration. The fact that the integral of a quantity is indicated by the area under its graph is very useful in obtaining numerical values of definite integrals (*a*) in cases where a graph of a function is given but no formula is available and (*b*) in cases where the formula is available but it is easier or faster to prepare a graph than to calculate the integral. Some common methods of obtaining the area under a graph, and thereby the integral of a function, will be mentioned.

a. Square counting. If we are given a graph of a function on rectangular coordinate paper, as in Fig. 10-9, we can obtain the area beneath the graph by counting the squares enclosed by the graph, the horizontal axis, and the vertical lines indicating the limits of integration. After counting all the included squares which are not cut by the graph itself, we can inspect the squares which are not wholly included. We estimate what fraction of each square is included. In the figure, for instance, the area beneath the curve is found to include about 31.2 squares.

b. Finding the mean ordinate. In accordance with Sec. 10-2, if we find the average height (mean ordinate) of a graph and multiply it by the length of the interval in question, we should have the area under the

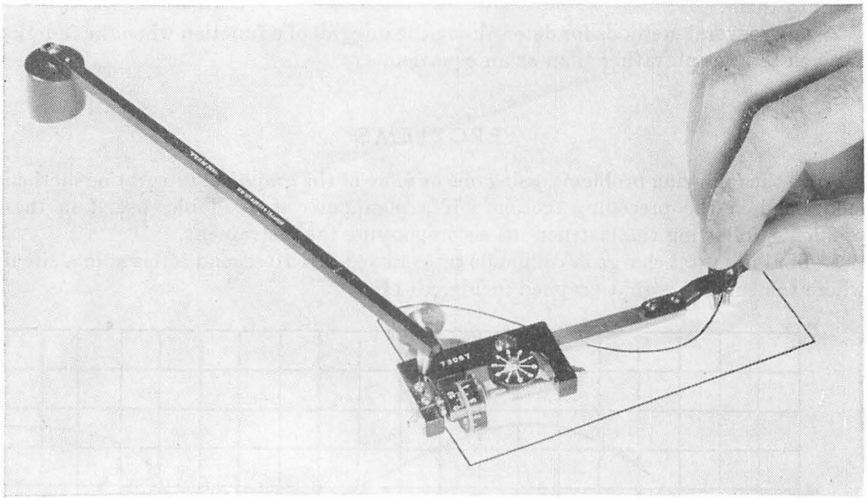


Fig. 10-10

graph. We can estimate the average height of the graph by measuring the height at several equally spaced points along the horizontal axis and taking the average of these values. You should check that this procedure, applied to Fig. 10-9, gives a result similar to that obtained by square counting.

c. Area rules. Certain rules have been devised which, when applied to irregular curves, give with varying degrees of accuracy the areas within the curves.[3,4]

d. Weighing. A procedure sometimes resorted to is first to weigh the rectangular sheet of paper upon which the graph is drawn by means of a delicate chemical balance. The desired area is then cut out with scissors and weighed. Comparison with the original weight of the rectangle establishes the area beneath the irregular graph.

e. The planimeter. Perhaps as elegant a method as any for obtaining mechanically the area of a figure is the use of a planimeter, an important engineering instrument. One form of planimeter is shown in Fig. 10-10. When the point of the stylus of this device is moved carefully around the perimeter of an irregular figure, the indicating dials record the area of the figure. An instrument of this kind is a "must" in many engineering projects. The theory of the planimeter is presented in more advanced courses.

QUESTIONS

1. Name several instruments whose indications are the integrals of the functions fed into them.
2. Give several methods for determining the integral of a function when the function is given as a graph rather than as an equation.

PROBLEMS

Solve the following problems, using one or more of the graphical integration methods mentioned in the preceding section. If a planimeter is available, use it in these problems, following the instructions accompanying the instrument.

1. Find the total charge in coulombs transmitted in a 10-second interval in a circuit where the current is that graphed in Fig. 10-11.

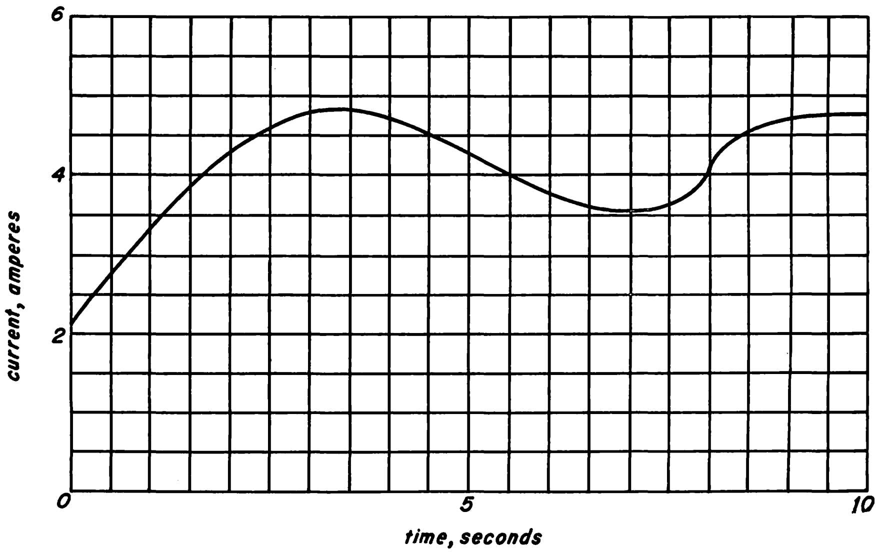


Fig. 10-11

2. The power in a circuit varied as shown in Fig. 10-12. (*a*) Find the total energy delivered over a 5-hour interval as shown. (*b*) Find the average power over this interval.

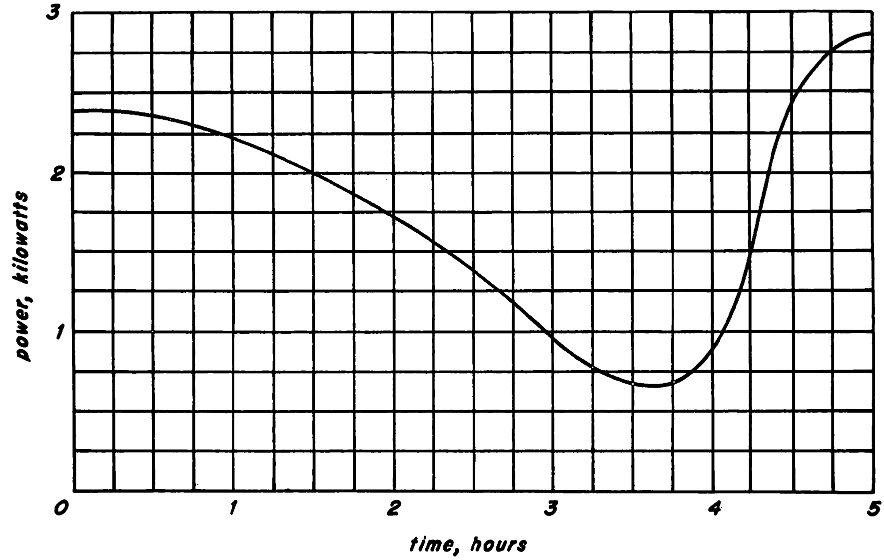


Fig. 10-12

3. The elevation of the terrain above sea level, along a radial from the antenna of a television station, varied as shown in Fig. 10-13. Find the average height of the antenna above the terrain along this radial over the interval from 2 to 10 miles from the antenna. (Note that the horizontal and the vertical scales are not the same.)

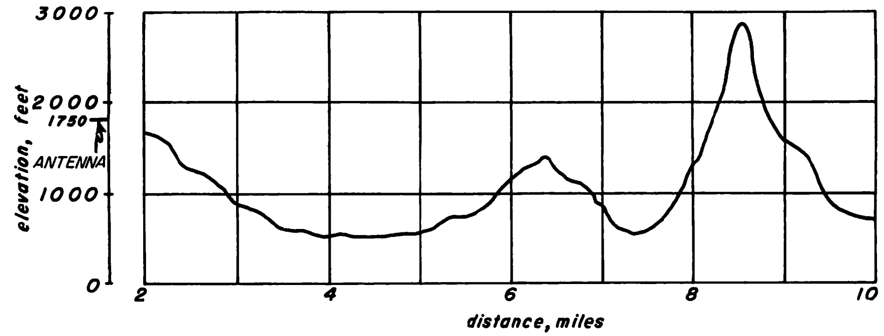


Fig. 10-13

4. Two transformer cores, similar except for the materials used, when tested under similar conditions gave hysteresis loops as shown in Fig. 10-14. Which core (if either) caused the greater hysteresis loss and in what ratio to the loss in the other core? (The loss is proportional to the area of the figure.)

5. The horizontal-plane radiation pattern of a certain antenna, in millivolts per meter at 1 mile, neglecting attenuation, was that shown in Fig. 10-15 when the power input was 1 kilowatt. Find the rms value of this radiation pattern. (Find the radius of a circular pattern having the area of the given pattern.)

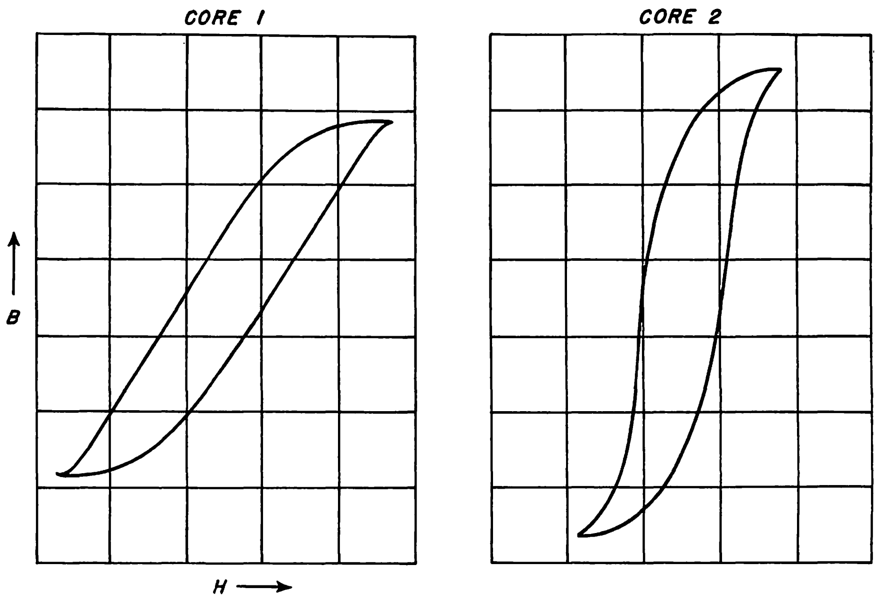


Fig. 10-14

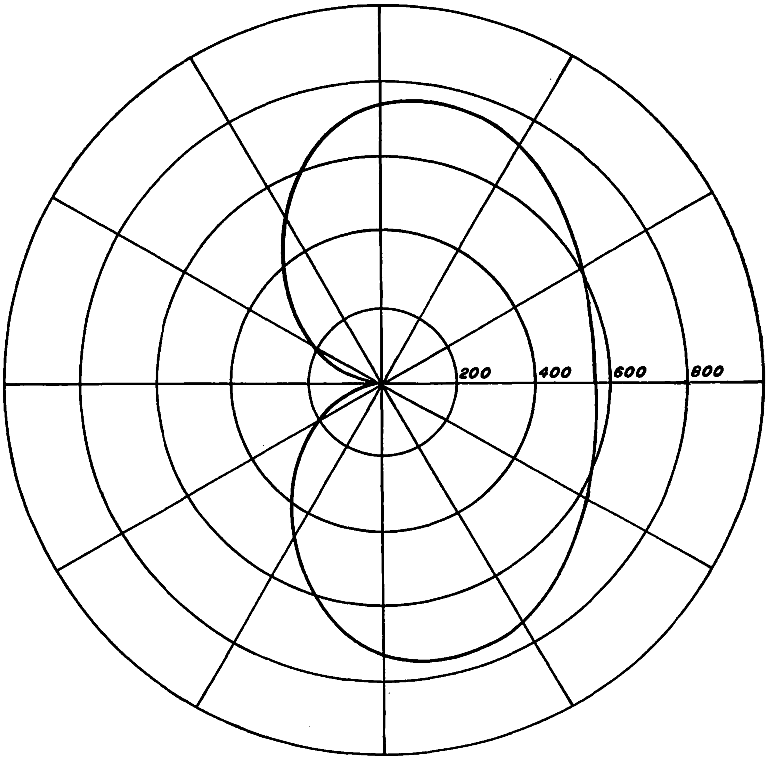


Fig. 10-15

10-9 Conclusion. Traditionally, mathematics of derivatives and differentials is called *differential calculus*, while that of integration and related problems is called *integral calculus*. Together, these make up the science of *infinitesimal calculus*, or simply *calculus* (differentials were once considered as being very small, approaching zero; that is, they were infinitesimal).

The honor of the invention of calculus is shared by Sir Isaac Newton (1642–1727) of England and Gottfried Leibnitz (1646–1716) of Germany, who worked independently. Newton's early discoveries of calculus occurred a decade before those of Leibnitz, but Leibnitz published his first. There followed an unfortunate controversy among followers of these great men as to whether plagiarism was involved in Leibnitz's publications. Today, both men are honored for the discovery, although certain of their concepts were confused because they lacked the exact *limit* idea employed today.

REFERENCES

1. D. Bierens de Haan: "Nouvelles tables d'intégrales définies," P. Engels, Leyden, 1867; reprinted, Hafner Publishing Company, New York, 1939.
2. C. F. Lindman: "Examen des nouvelles tables d'intégrales définies de M. Bierens de Haan," *Kgl. Svenska Vetenskapsakad. Handl.* (Stockholm), **24**(5) (1891); reprinted, Hafner Publishing Company, New York, 1944.
3. C. D. Hodgman: "Mathematical Tables from Handbook of Chemistry and Physics," 9th ed., p. 295, Chemical Rubber Publishing Co., Cleveland, Ohio, 1948.
4. R. S. Burington: "Handbook of Mathematical Tables and Formulas," 3d ed., p. 13, Handbook Publishers, Inc., Sandusky, Ohio, 1949.

Part Three

ADDITIONAL FUNCTIONS

11

Trigonometric Functions

The preceding portions of this book have dealt mainly with power functions, for example, $y = 2x^3 + 12x$. We now take up *trigonometric functions* (or *circular* functions), for instance, $y = \sin x$ or $y = \sec x \tan x$. Such functions play a major role in electrical work, particularly in ac studies.

11-1 Polar coordinates. In studying many functions it is convenient to use *polar coordinates* to specify the location of a point. Figure 11-1 illustrates the relationship between polar coordinates and the rectangular coordinates which we have used up to this point. In diagram (*a*) the location of point P is given in rectangular coordinates by showing the x and y distances which identify the point. In general, a point P whose coordinates are x and y is referred to as the point $P(x,y)$. Specifically, if the coordinates of the point are $x = 3$ and $y = 4$, we identify it by $P(3,4)$.

In diagram (*b*) the same point P is identified by giving (1) the length r of the straight line called the *radius vector* which connects the given point P with a reference point O called the *pole*, and (2) the angle θ by which r is displaced from the *polar axis*, a horizontal line OX. We

refer to r and θ as the *polar coordinates** of P, and P may be referred to as the point $P(r,\theta)$.

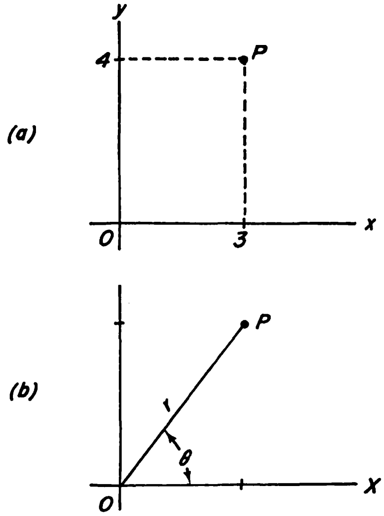


Fig. 11-1

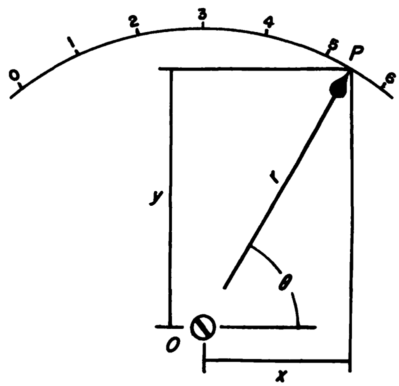


Fig. 11-2

By well-known right-triangle relations we easily get these important equations relating polar and rectangular coordinates designating a point in a plane:

$$x = r \cos \theta \qquad y = r \sin \theta \qquad r^2 = x^2 + y^2 \qquad \tan \theta = \frac{y}{x} \tag{1}$$

Polar coordinates might be applied to the study of the motion of the instrument pointer in Fig. 11-2. It is true that we could indicate the location of the tip P of the pointer by giving in rectangular coordinates the horizontal and vertical distances x and y from a reference point, such as the pivot O. But it is more convenient for many purposes to state the position of P in polar coordinates, giving the length r of the pointer and the angle θ which it makes with the horizontal.

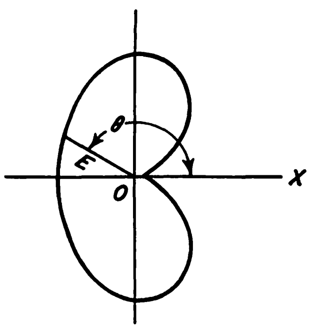


Fig. 11-3

Polar coordinates are useful in expressing *functional relationships* between two quantities. The angle θ is often taken as the independent variable (sometimes called the *argument*), while r may be the dependent variable (called the *modulus*). Figure 11-3, for example, shows how the rf field intensity **E** (indicated

* The radius vector r is also often indicated by ρ. In some treatments r is allowed to take both positive and negative values. Here, we assume that r is always positive.

by the length of the radius vector) at a certain distance from an antenna might vary as a function of the angle θ.

11-2 Projections. In the work which follows it will be convenient to use the idea of the *projection* of one line upon another. Referring to Fig. 11-4, the projection of a line segment of length s upon a second line AB is defined as the distance p between perpendiculars dropped from the

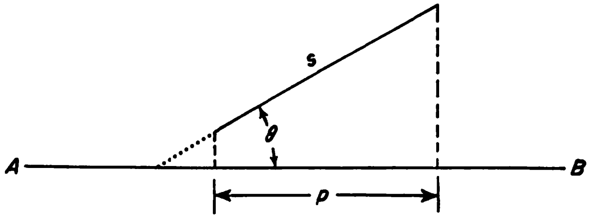


Fig. 11-4

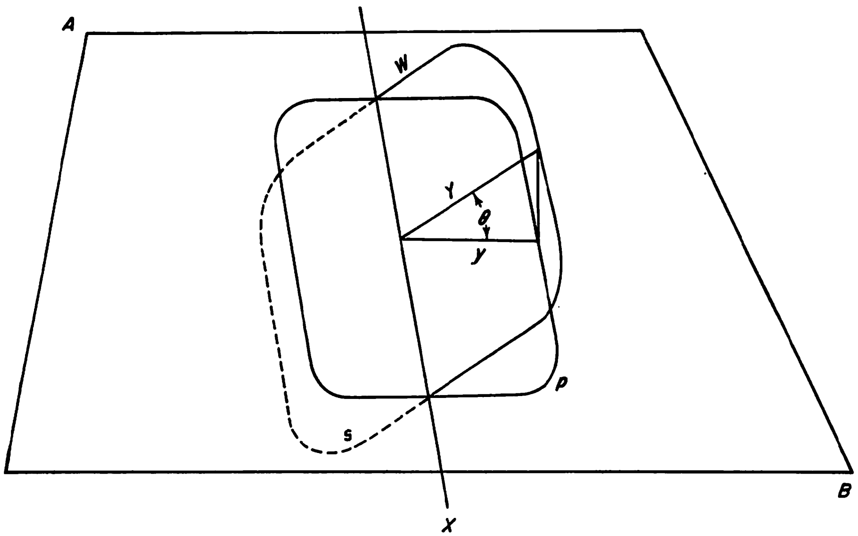


Fig. 11-5

ends of s upon AB. Let θ be the angle between s (extended) and AB. We have $\cos\theta = p/s$, or

$$p = s \cos \theta \qquad (2)$$

which enables us to calculate the length of the projection p.

If s is the area of a plane figure, rather than the length of a line, its projection p upon another plane AB is taken as that area marked off on AB by perpendiculars dropped from *each point* on the perimeter of the given figure. We can use (2) to calculate the projected area p. In Fig. 11-5, for instance, the coil W is situated in a uniform vertical mag-

netic field. Let its cross-sectional area be s. The number of flux lines which pass through W is equal to the number of lines passing through its projected area upon the horizontal plane AB. To show that (2) gives this area, we note that the area of the coil is given by

$$s = 2\int Y\,dx$$

from Chap. 9. And the area p of the projection of s upon AB is

$$p = 2\int y\,dx$$

But y is everywhere the projection of some line Y in the original coil area, so that $y = Y \cos \theta$. This gives

$$p = 2 \cos \theta \int Y\,dx$$

or

$$p = s \cos \theta \qquad (2)$$

QUESTIONS

1. What two quantities are needed to fix the position of a point in a plane using rectangular coordinates? Using polar coordinates?
2. Given the x and y coordinates of a point in a plane, what formulas would we use to transform this information into polar coordinates?
3. What formulas would transform the position of a point in a plane, expressed in polar coordinates, into rectangular coordinates?
4. Define the *projection* of a line segment upon a second line. What formula gives the length of this projection?

PROBLEMS

1. In a directional broadcast antenna, tower 2 is located 212 feet from tower 1, in a direction 38° north of east. How far to the north of tower 1 is tower 2 situated?

2. A target-practice object is observed on a radar at an airline distance of 18,000 feet and at an elevation angle of 72°. If the object were shot down, at what horizontal distance from the observer would the wreckage fall, assuming a vertical fall?

3. A radar screen shows an object 70 miles from the observer at an angle of 50° east of north. How far to the east of the observer is this object? (The reference axis here is in a "vertical," or northerly, direction on the paper.)

4. In Fig. 11-6 find the projection of the line CD upon line AB. (Note that this is the same as the projection of EF upon AB.)

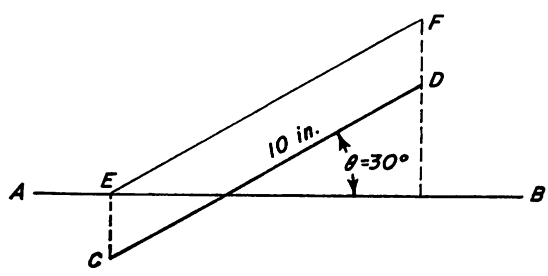


Fig. 11-6

5. A microphone diaphragm intercepts 6.75×10^{-9} watt of acoustic power when turned broadside to a sound source. What will be the (theoretical) intercepted power if the diaphragm is turned at an angle of 55° to the source?

6. Light radiations having a plane wavefront strike a photosensitive surface at an angle of 30°. If the surface were turned to face the light directly, by what factor would the amount of received light energy be increased? (The surface is assumed smaller than the cross section of the light beam.)

7. A *curtain* receiving antenna is broadside to a distant transmitter. If it is now turned through an angle of 21°, by what factor will the radiated power impinging upon the curtain be reduced? (This calculation does not include the effect of the antenna directional pattern.)

8. The unattenuated field intensity of the radiation from a certain antenna at a distance of 1 mile varied approximately according to $\mathbf{E} = 1{,}800(1 - \cos\theta)$ millivolts per meter. On polar graph paper plot a graph of the radiation pattern of the antenna.

9. On polar-coordinate paper plot a graph of the function $r = 10 \sin 2\theta$.

10. A half-wave dipole has a radiation pattern given by $\mathbf{E} = K \cos[(\pi/2)\cos\theta]$. Plot this pattern on polar-coordinate paper if $K = 50$.

11-3 Radian measure.

For *numerical calculations* it is most convenient to consider angles as measured in degrees and decimal parts of a degree. For *calculus operations*, however, a different unit is handier. This unit is called the *radian*. In radian measure, an angle θ is expressed as the ratio of the circular arc s which it intercepts to the radius r of the circle. That is,

$$\theta = \frac{s}{r} \qquad \text{radians} \tag{3}$$

Accordingly,

An angle of one radian is that angle subtended (cut off) at the center of a circle by an arc having a length equal to the radius.

Figure 11-7 shows a circle of radius r with its center at O. The angle θ ($= \angle AOB$) is here of such size that the length of the arc AB is equal to the radius OA. Here, then, $\theta = 1$ radian.

We must note that an angle of 1 radian does *not* cut off a chord equal to r. For example, if we should draw a straight line (chord) connecting A and B in the figure, the length of this line would not be equal to the radius r.

The *relation between radian measure and degree measure* is easily found. Since the circumference of a circle is $C = 2\pi r$, then 2π radians are required to make up one complete rotation of r. Thus 2π radians $= 360°$, or 1 radian $= 57°17'44.8''$, approximately. For most purposes, the following approximations are sufficient:

$$1 \text{ radian} = 57.3° \qquad 1° = 0.01745 \text{ radian*} \tag{4}$$

* It is recommended that the word "radian" not be abbreviated. Sometimes expressions like 2 rdn or $2^{(r)}$ are found, indicating two radians, but they are exceptions. Radian measure, incidentally, is also called *circular measure.*

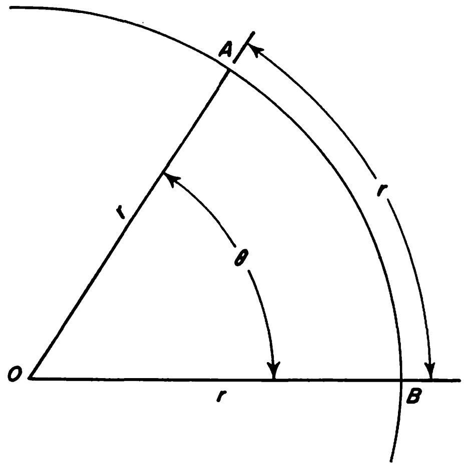


Fig. 11-7

11-4 Angular speed and acceleration. In Fig. 11-8 consider the point P as rotating about the center O, so that the angle θ changes with time. The *angular speed* of P at any time t is defined as the rate of change of θ. This quantity is represented by ω, so that

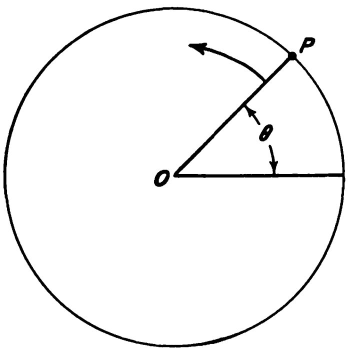


Fig. 11-8

$$\omega = \frac{d\theta}{dt} \tag{5}$$

We shall measure θ usually in radians and ω in radians per second.

If ω changes with respect to time, we represent the rate of change of angular speed by the term *angular acceleration*, indicated by α:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{6}$$

11-5 Relations of angular quantities to linear. Let Fig. 11-9 represent a rotating object, such as an armature. The armature, taken as a unit, possesses *angular* motion, but at the same time, different points on the armature (such as P_1 and P_2) are moving along circles. Since P_2 is situated closer to the rim than P_1 is, we find P_2 traveling along a larger circle than P_1 does. Thus in a given time P_2 describes a greater linear distance than P_1 does.

To establish the relation between the angle traversed by P_2 and the linear distance through which it travels, we use (3), getting $s_2 = r_2\,\theta$. Thus, in general, any rotating point P covers a linear distance

$$s = r\theta \tag{7}$$

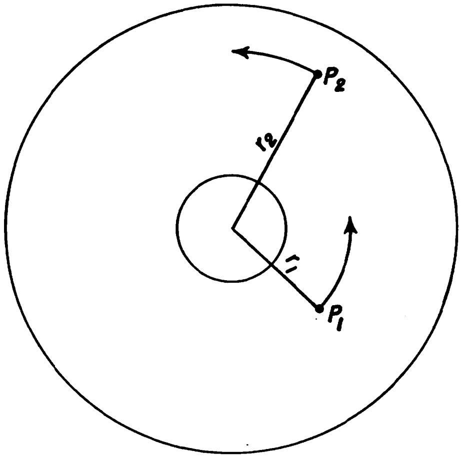


Fig. 11-9

An equation for the linear speed of the point P at any instant, in terms of the angular speed, is obtained by differentiating (7):

$$\frac{ds}{dt} = r\frac{d\theta}{dt} \quad \text{or} \quad \mathbf{v} = r\omega \tag{8}$$

A further differentiation gives the linear acceleration of P:

$$\frac{d\mathbf{v}}{dt} = r\frac{d\omega}{dt} \quad \text{or} \quad \mathbf{a} = r\alpha \tag{9}$$

11-6 Component velocities. When a point P moves along a curved path, its *direction* is, at any instant,taken to be *along a tangent* to the curve.

In Fig. 11-10, for instance, the point P moves along a circle at a fixed distance r from the center O. Let P move in such a way that the rate of change of the angle θ is fixed, that is, let ω be a constant. Let us now find the rates of change of the *rectangular* coordinates x and y of point P.

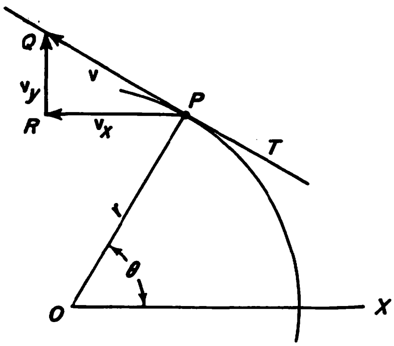


Fig. 11-10

At the instant depicted, P is moving along the tangent line T. Its speed, by (8) is $\mathbf{v} = r\omega$. Let us indicate the velocity (speed and direction) of P, at this instant, by the vector PQ. This vector can be resolved into horizontal and vertical components $\mathbf{v}_x = dx/dt$ and $\mathbf{v}_y = dy/dt$, as shown. (That is, the vector sum of $\mathbf{v}_x$ and $\mathbf{v}_y$ is the actual velocity $\mathbf{v}$.)

Since a tangent to a circle at any point P is perpendicular to the radius at P, we have the fact that T is perpendicular to r. Also, QR is perpendicular to the polar axis OX. Therefore the angle PQR is equal to the central angle θ, for its two sides are perpendicular, respectively, to the sides of θ. From the figure,

$$\mathbf{v}_x = \frac{dx}{dt} = r\omega \sin \theta \qquad \text{and} \qquad \mathbf{v}_y = \frac{dy}{dt} = r\omega \cos \theta \tag{10}$$

Example. A directional antenna is mounted on a circular platform whose radius is 5 feet. The platform is turned counterclockwise at 1 revolution per minute. When a point P on the circumference is 60° north of east from the center, find (*a*) the velocity of P toward the west and (*b*) the velocity of P toward the north.

The angular velocity of the platform is ω = 1 revolution per minute = $2\pi/60 = \pi/30$ radian per second. By (10), we find (*a*) the velocity to the west is $\mathbf{v}_x = 5(\pi/30) \sin 60° = (3.142/6)0.866 = 0.453$ foot per second, and (*b*) the velocity to the north is $\mathbf{v}_y = 5(\pi/30) \cos 60° = (3.142/6)0.5 = 0.262$ foot per second.

QUESTIONS

1. Give a formula expressing an angle θ in radians in terms of an intercepted circular arc and the radius of the arc.
2. How many degrees in 1 radian? How many radians in 1 degree?
3. If a point P rotates about a center O, what formula expresses its angular speed?
4. If a point P rotates about a center O, what formula expresses its angular acceleration?
5. If a point P rotates about a center O, give a formula for the distance s traveled as P traverses an angle θ.
6. As a point P rotates about a center O, what expressions give (*a*) its linear speed and (*b*) its linear acceleration?
7. Give formulas for the components of velocity of a point (*a*) in the horizontal, or x, direction and (*b*) in the vertical, or y, direction as the point traverses a growing angle θ about a center O.

PROBLEMS

1. How many radians correspond to each of the following angles? Express as multiples of π radians.

(*a*) 180°	(*c*) 45°	(*e*) 30°	(*g*) 20°
(*b*) 90°	(*d*) 60°	(*f*) 15°	(*h*) 54°

2. How many degrees correspond to each of the following angles expressed in radians?

(*a*) $\pi/4$	(*c*) $\pi/9$	(*e*) $5\pi/3$	(*g*) $4\pi/3$
(*b*) $3\pi/2$	(*d*) $2\pi/3$	(*f*) $\pi/10$	(*h*) $3\pi/4$

3. An instrument pointer moves through an arc of 270°. To how many radians is this equivalent?

4. The radiation pattern of an antenna has a minimum value in a direction 18° off the antenna axis. Express this angle in radians.

5. An armature turns at 1,800 revolutions per minute. To what value of ω, in radians per second, does this correspond?

6. The coil of an instrument rotates at the rate of 0.005 radian per millisecond. Express this angular speed in degrees per second.

7. A motor accelerates at a rate of 600 revolutions per minute per second. To how many radians per second per second is this equal?

8. An instrument pointer is 2.1 inches long. The tip of the pointer moves over a scale 2.4 inches long. What angle does the pointer describe, in radians?

9. An alternator has a rotating field which is 32 inches in diameter. When the field is turned at 120 revolutions per minute, what is the linear speed of a point on its circumference?

10. If the field assembly of Prob. 9 is accelerated at 12 revolutions per minute per second, what linear acceleration is applied to a point on its circumference?

11. If the field assembly of Prob. 9 turns in a counterclockwise direction, what is the upward component of the velocity at a point P on its circumference when P is at an angle of 45° above the horizontal?

12. An airplane circles a point at a distance of 3 miles, taking 3 minutes for each counterclockwise turn. When the plane is at an angle of 30° north of east, what is the westward component of its velocity in feet per second?

13. It can be shown that, when an armature of radius r rotates at ω radians per second, a point on its circumference is given a constant normal acceleration toward the center equal to $\mathbf{a}_n = r\omega^2$. If an armature 0.3 meter in diameter is rotated at 2,000 revolutions per minute, to what normal acceleration will a conductor on the surface be subjected? Using $\mathbf{F} = m\mathbf{a}$, what centrifugal force in newtons will be applied to a conductor of mass 0.04 kilogram located at the circumference?

14. A lever in a magnetic switch turns, during a certain interval, at a rate equivalent to $t - 2t^2$ revolutions per second. (*a*) Write an equation for the angular speed ω, in radians per second, of the lever. (*b*) Find an equation for the angle θ through which the lever moves in a given time. (*c*) Find an equation for the angular acceleration α of the lever at any time. (*d*) If the length of the lever is 4 inches, and if it is pivoted at one end, find the distance s through which the moving end has traveled after 0.3 second. (*e*) Find the linear speed $\mathbf{v}$ of the moving end when $t = 0.3$ second. (*f*) Find the linear acceleration $\mathbf{a}$ of the moving end when $t = 0.3$ second.

11-7 Derivative of the sine function.

In Fig. 11-11 we find the point $P(x,y)$ moving along a circle, describing an angle which we shall call u radians at the center of the circle. We proceed to find a formula for the rate of change of the sine of u as u is varied. Although this is accomplished by the delta method in more formal treatments, the following presentation illustrates the result. By (10),

$$\frac{dy}{dt} = r\omega \cos u$$

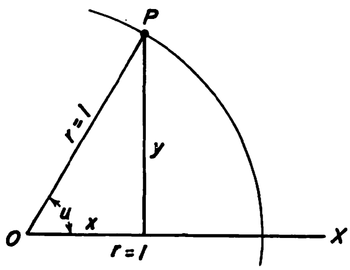


Fig. 11-11

It will be convenient to consider a special case in which r has a length equal to 1

unit. We note that this makes $y = \sin u$, so that in the present case the above equation becomes

$$\frac{d}{dt} \sin u = \omega \cos u \tag{11}$$

We may write du/dt for ω:

$$\frac{d}{dt} \sin u = \cos u \frac{du}{dt} \tag{12}$$

We may multiply both sides of (12) by dt/dx:

$$\frac{d}{dx} \sin u = \cos u \frac{du}{dx} \tag{13}$$

We have, then, the interesting and simple result that

The derivative of the sine of an angle is equal to the cosine of that same angle times the derivative of the angle.

In (13), x may be *any* variable of which u is a function and is not limited to the variable x in the figure.

11-8 Derivative of the cosine function. To find the derivative of the cosine of an angle u we make use of the familiar trigonometric identity*

$$\sin^2 \theta + \cos^2 \theta = 1$$

from which

$$\cos^2 u = 1 - \sin^2 u$$

Differentiating implicitly,

$$2 \cos u \frac{d}{dx} \cos u = -2 \sin u \frac{d}{dx} \sin u$$

By (13), the final factor is equal to $\cos u \, du/dx$. The above equation simplifies immediately to

$$\frac{d}{dx} \cos u = - \sin u \frac{du}{dx} \tag{14}$$

In words,

The derivative of the cosine of an angle is equal to minus the sine of the angle times the derivative of the angle.

Example 1. The rectangular coil of Fig. 11-12 turns in a uniform magnetic field. Let Φ_{max} represent the amount of flux passing through the coil when it is in a horizontal position. When the coil has turned through an angle θ, the flux passing through it is smaller, being proportional to the projection of the coil

* In what follows, it is sometimes necessary to use formulas called *trigonometric identities*, which are ordinarily presented in courses in trigonometry. Certain of these identities are listed in Table 1 of the Appendix.

upon the horizontal plane $X'X$. Thus for any angle θ, the flux is

$$\phi = \Phi_{max} \cos \theta \tag{A}$$

Let the coil turn at a constant rate ω radians per second, so that $\theta = \omega t$. Then

$$\phi = \Phi_{max} \cos \omega t \tag{B}$$

Differentiating (B) according to (14),

$$\frac{d\phi}{dt} = -\omega\Phi_{max} \sin \omega t \tag{C}$$

Recalling that the induced emf in a coil is $v_{ind} = -N\, d\phi/dt$, we get from (C)

$$v_{ind} = \omega N \Phi_{max} \sin \omega t \tag{15}$$

That is, a *sine wave* of voltage is induced in the coil as a result of its rotation in the magnetic field. We note that the greatest value which can be attained by $\sin \omega t$ is 1; hence we can determine from (15) that the greatest value of the induced emf is

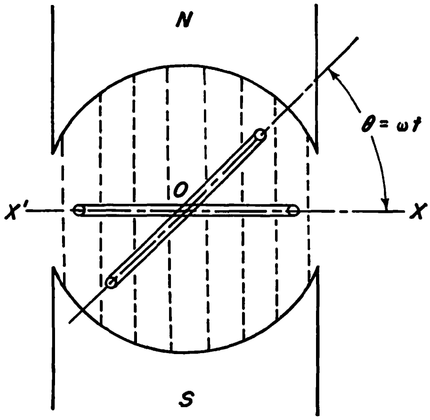


Fig. 11-12

$$V_{max} = \omega N \Phi_{max}$$

Observe that to make $\sin \omega t$ equal to 1, and thus obtain a maximum of induced emf, the coil must have turned through an angle $\omega t = \pi/2$, or $3\pi/2$, etc. Interestingly enough, the actual value of ϕ at these points is, by (B), equal to zero.

Example 2. Consider the circuit of Fig. 11-13a. The impedance of this

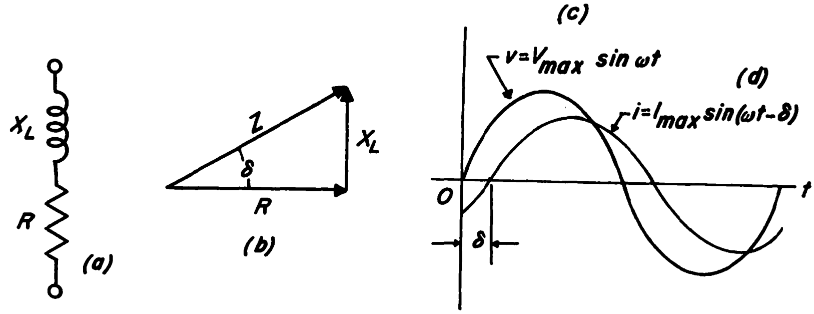


Fig. 11-13

circuit has a phase angle δ, as illustrated in the vector diagram (b) in the figure. Assume that there is applied to this circuit a voltage wave of the form $v = V_{max} \sin \omega t$, which is graphed in diagram (c). Find di/dt.

From our knowledge of elementary electricity, we see that the current i will be different in phase by an angle $-\delta$ from the applied voltage. The current wave is shown in diagram (d). Had the current been *in phase* with the voltage, its

equation would have been $i = I_{\max} \sin \omega t$; but this form must be modified to allow for the phase angle δ, so that the actual current equation is $i = I_{\max} \sin(\omega t - \delta)$. Differentiating the current wave according to (13), we have

$$\frac{di}{dt} = I_{\max}[\cos(\omega t - \delta)]\omega = \omega I_{\max} \cos(\omega t - \delta)$$

Example 3. Differentiate $y = \cos^3 x$.

This function is treated primarily as a power function, since the quantity $\cos^3 x$ is actually $\cos x$ raised to the third power. (Letting $\cos x$ be called u, we could regard the function as being $y = u^3$ and differentiate accordingly.) We have, then,

$$\frac{dy}{dx} = 3\cos^2 x \frac{d}{dx}\cos x = -3 \sin x \cos^2 x$$

PROBLEMS

In Probs. 1 to 12 differentiate with respect to the independent variable x or t.

1. $y = \sin 2x$
2. $y = 3 \sin x$
3. $y = 12 \sin 14t$
4. $y = 169.2 \sin 377t$
5. $y = \sin t^2$
6. $y = 2 \cos 3t^3$
7. $y = 1{,}000 \cos(t^2 - t^3)$
8. $y = 10t^3 + \cos t$
9. $y = \sin^2 t$
10. $y = -\cos^2 t$
11. $y = 2 \sin^2 t^2$
12. $y = \sin^2 t - 2 \cos 2t^2$

13. Let the primary current in a transformer be $i_1 = I_{\max} \sin \omega t$, where $I_{\max}$ is the crest value of the current. Write a formula for the induced secondary emf v_2.

14. In Prob. 13, state a formula for the maximum value of v_2. For what value of i_1 will the greatest value of v_2 occur?

15. A voltage $v = 2{,}000 \sin 500t$ is impressed across a 20-microfarad capacitor. Find a formula for the resulting current.

16. In Example 1 of Sec. 11-8, suppose the coil has 1,000 turns and that it rotates at 3,000 revolutions per minute. The maximum flux through the coil is 6×10^{-4} weber. (*a*) What equation expresses the induced emf? (*b*) What is the induced emf when $t = 0.0176$ second? (*c*) What is the rate of change of induced emf at that time?

17. Experiment shows that the current $I_{\max}$ at a current loop in a vertical antenna of length l is related to the base current by $I_{\max}/I_{\text{base}} = 1/\sin(2\pi l/\lambda)$. How fast does this ratio change with respect to the transmitted wavelength λ when l is constant?

18. In televising a poster of area A, a light source of intensity I candles is used at a distance r from the poster. The luminous flux F lumens supplied to the poster is $F = IA(\cos\theta)/r^2$, where θ is the angle between the incident rays and a normal to the poster surface. How fast does F change with respect to θ?

19. A rectangular coil of length l and width $2r$ is made up of N turns. When the coil is inserted into a magnetic field of density **B** and a current I is sent through the coil, the resulting torque applied to the coil is $\mathbf{T} = 2\mathbf{B}lINr \cos\theta$, where θ is the angle between the plane of the coil and the direction of the flux. What expression gives the rate of change of **T** as the coil rotates?

20. At a certain point distant from a transmitting antenna the intensity **E** of the electric field associated with the transmitted wave varies with distance y as follows:

$\mathbf{E} = \mathbf{E}_{max} \sin (2\pi y/\lambda)$. Here $\mathbf{E}_{max}$ is the crest value of the field, and λ is the wavelength. Find the rate at which $\mathbf{E}$ varies with y at a given instant.

21. The current in a 30-ohm resistor is $i = 12 \sin 118t$. How fast is the power in the resistor changing at any instant?

22. A single-turn coil of radius r carries a current I. A point P is located on the coil axis at a distance s from the coil. Then a line PQ connecting P with any point Q on the coil circumference makes an angle θ with the coil axis. It can be shown that the magnetic-field intensity at P due to I is $\mathbf{H} = (2\pi I/r) \sin^3 \theta$. Get a formula for the rate of change of $\mathbf{H}$ with respect to θ.

23. A carrier wave of amplitude V_0 and an angular frequency ω_c is modulated by a sinusoidal signal of angular frequency ω_s. The degree of modulation is m. Under these conditions, the modulated wave has the equation $v = V_0(1 + m \sin \omega_s t) \sin \omega_c t$. Obtain an expression for the rate of change of v.

24. The field intensity at a distance r from an isolated straight-wire transmitting antenna whose length is an integral odd number N of half wavelengths is $\mathbf{E} = (60I/r)[\cos (2\pi N \cos \theta)]/\sin \theta$, where I is the antenna current and r is taken at an angle θ with respect to the antenna axis. Find a formula for the rate of change of $\mathbf{E}$ with respect to θ.

11-9 Simple harmonic motion. The term *simple harmonic motion* is used to describe a kind of motion which is encountered in many devices of interest in electricity. To describe this motion we use Fig. 11-14.

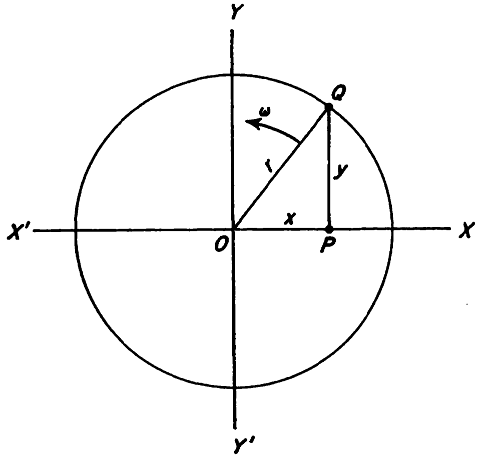


Fig. 11-14

Let the point Q move at a constant speed $\mathbf{v}_Q$ along the circle shown, and let its corresponding angular speed be called ω. Then, at any time t, the point Q will have described an angle ωt, and in accord with (1) the location of Q will be given by

$$x = r \cos \omega t \qquad y = r \sin \omega t \tag{16}$$

where r is the radius of the circle. Let OP be the projection of r upon the x axis. Then as Q moves around the circle, P moves back and forth

along the x axis, having a displacement from the center O always given by the first equation in (16). The velocity of P can be obtained by differentiating that equation:

$$\mathbf{v} = \frac{dx}{dt} = -\omega r \sin \omega t \tag{17}$$

or

$$\mathbf{v} = -\omega y \tag{18}$$

In many cases we need to find the velocity of P directly in terms of its displacement x. The desired relation is found by noting that $y = (r^2 - x^2)^{1/2}$, so that (18) can be written

$$\mathbf{v} = -\omega(r^2 - x^2)^{1/2} \tag{19}$$

Further, we may differentiate (17), getting the acceleration of P:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\omega^2 r \cos \omega t \tag{20}$$

or

$$\mathbf{a} = -\omega^2 x \tag{21}$$

This latter equation is typical of simple harmonic motion. In fact,

Simple harmonic motion is defined as motion of a point in a straight line such that its acceleration is always proportional to its displacement from a center O and always in a direction opposite to its displacement.

In (21) the minus sign indicates that the acceleration is actually in a direction opposite to that of the displacement x of point P, and we say that P has simple harmonic motion. A more general expression, not restricted to the construction of Fig. 11-14, would be the statement that a point has simple harmonic motion if its acceleration is always

$$\mathbf{a} = \frac{d^2x}{dt^2} = -kx \tag{22}$$

where k is any positive constant. We take (22) as the defining equation of simple harmonic motion. In the arrangement of Fig. 11-14 k has the particular value ω^2.

The time T required for 1 cycle of the motion of the point, that is, for one complete set of changes in position, is called the *period* of the motion. The maximum displacement of a point resulting from simple harmonic motion is the *amplitude* of the motion. In Fig. 11-14 this is equal to r. It will be noted from the preceding equations that, when a point has simple harmonic motion, its displacement, velocity, and acceleration have graphs, with respect to time, which are of the general form of a sine curve, although they are displaced from each other along the time axis.

Example. A generator for a transmitting plant is driven by an engine which operates at 600 revolutions per minute. The stroke is 18 inches. Assuming that the piston has approximately simple harmonic motion, what is the velocity of the piston when it is 4 inches from the end of its stroke? What is its acceleration then?

Take the center of the stroke as a reference. The amplitude of the motion of the piston is $r = 9$ inches. When the piston is 4 inches from the end of its stroke, its displacement from the reference position is $x = 5$ inches. We observe that $\omega = 600$ revolutions per minute $= 62.83$ radians per second. These substitutions in (19) give

$$\mathbf{v} = -62.83(9^2 - 5^2)^{1/2} = -470.2 \text{ inches per second}$$

The acceleration is found by (21):

$$\mathbf{a} = -(62.83)^2(5) = -19{,}738 \text{ inches per second per second}$$

PROBLEMS

1. Draw a figure similar to Fig. 11-14, but let P_1 be the projection of Q upon the (vertical) y axis. Then as Q moves along the circle, $y = r \sin \omega t$. Show that P_1 has simple harmonic motion.

2. By successive differentiations show that motions described by these equations are simple harmonic motions:

(a) $x = 5 \sin (\omega t + \pi)$

(b) $y = 2 \cos (\omega t + \pi/6)$

(c) $y = 10 + 6 \cos (\omega t + 1)$

(d) $x = 244 + \sin (\pi/4) \cos (\omega t + \pi)$

3. In a device for mechanically determining the directional pattern of an antenna the end P of a lever has a displacement equal to $K \cos \theta$, where θ is the angle described by a bar rotating uniformly about one of its ends. If the bar makes 1 revolution in 40 seconds, find the greatest acceleration of P in terms of the constant K.

4. A shear is used to cut to size some parts used in electron tubes. The blade executes approximate simple harmonic motion. The amplitude is 2 centimeters, and the period is 0.1 second. Find the acceleration and the speed of the blade when its displacement is 0.7 centimeter.

5. A vibrator reed in a power supply makes 110 complete vibrations per second. The tip of the reed makes excursions of 0.2 centimeter each side of its rest position. Neglecting the curvature of the path and assuming simple harmonic motion, find the greatest acceleration of the reed tip.

6. An isolated radio station uses an accurate clock of the pendulum type as its time source. The pendulum bob swings each way precisely twice in 1 second, and its maximum excursion from its center position is 2.9 centimeters. Find the acceleration of the bob when it is 2 centimeters off center, assuming its motion to be in a straight line and of the simple harmonic variety.

7. An accurate 60-cycle tuning fork, associated with a carbon-mocrophone button, is used to control the frequency of a generating plant. The maximum excursion of the tip of the fork from its rest position is 0.8 millimeter. Calculate the maximum acceleration of the tip, assuming that the motion is in a straight line and of the simple harmonic variety.

8. An oscillograph draws upon a moving tape an inked graph of a sinusoidal current. The peak value of the current is 17.7 units, and the frequency is 21 cycles per second.

If 20 current units is indicated by a deflection of 1 centimeter, find the maximum acceleration in centimeters per second per second of the pen.

9. A direct current of 100 milliamperes is sent through a milliammeter having a range of 0 to 200 milliamperes. The length of the scale is 10 centimeters. Superimposed upon the direct current is an ac wave having the formula $i = 2.5 \sin 1.2\pi t$, where i is in milliamperes. Neglecting the curvature of the scale and assuming that the pointer accurately follows the current variations, find the acceleration of the pointer tip in centimeters per second per second when the ac component is at its peak.

10. A gasoline engine used to drive a dc generator turns at 3,000 revolutions per minute. Its stroke is 0.22 meter. The mass of a piston is 0.25 kilogram. Assume that the piston moves with approximately simple harmonic motion. Find the force due to acceleration upon the piston at the end of its stroke.

11-10 Further trigonometric derivatives. Having obtained equations for differentiating the sine and cosine functions, we easily get differentiation formulas for other trigonometric functions. To differentiate the tangent function, we write

$$\frac{d}{dx}\tan u = \frac{d}{dx}\frac{\sin u}{\cos u}$$

Differentiating according to the formula for a quotient,

$$\frac{d}{dx}\tan u = \frac{\cos u\, d(\sin u)/dx - \sin u\, d(\cos u)/dx}{\cos^2 u} = \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx}$$

or

$$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx} \tag{23}$$

By a similar procedure, we can obtain

$$\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx} \tag{24}$$

To get the derivative of sec u we write

$$\frac{d}{dx}\sec u = \frac{d}{dx}(\cos u)^{-1} = -\cos^{-2} u(-\sin u)\frac{du}{dx} = \frac{1}{\cos u}\frac{\sin u}{\cos u}\frac{du}{dx}$$

or

$$\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx} \tag{25}$$

Similarly

$$\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx} \tag{26}$$

QUESTIONS

1. State the formula for the derivative of the sine function; the formula for the derivative of the cosine function.

2. Define *simple harmonic motion.*
3. State an equation for simple harmonic motion.
4. State the formulas used in differentiating each of the following functions with respect to x: (*a*) $\tan u$, (*b*) $\cot u$, (*c*) $\sec u$, (d) $\csc u$.

PROBLEMS

In Probs. 1 to 16 differentiate with respect to x or t.

1. $y = \tan 5x$
2. $y = 2 \cot x$
3. $y = 3 \sec 2x$
4. $y = x^2 - \csc 2x$
5. $y = \tan^2 x + \cot 2x$
6. $y = 2 \tan x + x \sec x$
7. $y = \csc x - \cot x^2$
8. $y = x^2 \tan x + \csc^3 x^2$
9. $y = \sec^2 \omega t$
10. $y = \sec \omega t \tan \omega t$
11. $y = t^2 \csc \omega t$
12. $y = (t^2 - t) \csc (\omega t - \pi/2)$
13. $y = \csc \omega t \cot \omega t$
14. $y = \cos^2 \omega t \sec^{1/2} \omega t$
15. $y = t^2 \sec \omega t - \tan^2 \omega t$
16. $y = \sin (\cos \omega t + \sec \omega t)$

17. Derive Formula (24).

18. Derive Formula (26).

19. The dissipation factor D of an ac circuit is equal to the cotangent of the impedance phase angle θ. Find $dD/d\theta$.

20. The strength of a horizontal magnetic field may be measured as follows: let a compass needle (called a "magnetometer") be allowed to align itself with the known horizontal component $\mathbf{H}_h$ of the earth's field. If now the unknown field $\mathbf{H}_x$ is introduced at right angles to the earth's field, the needle will be deflected through an angle A. Then $\mathbf{H}_x = \mathbf{H}_h \tan A$. Find $d\mathbf{H}_x/dA$.

21. A formula useful in television gives the brightness of a surface in a given direction as $B = (I/a) \sec \theta$, where I is the intensity in the stated direction, a is the area of the surface, and θ is the angle between the normal to the surface and the stated direction. In case I can be taken as independent of θ, find $dB/d\theta$.

22. A certain aircraft radar antenna has a gain G which varies with vertical angle θ according to $G = G_0(\csc^2 \theta/\csc^2 \theta_0)$, where G_0 and θ_0 are constants. Find $dG/d\theta$.

23. In a tangent galvanometer, the current I is indicated by the deflection θ of the needle as compared with the deflection θ_a produced by a known current I_a, in accordance with $I = I_a \tan \theta/\tan \theta_a$. Noting that I_a and θ_a are constants, find $dI/d\theta$.

24. A given alternating current I_X flows in a fixed inductor, which is shunted by a variable resistor. The current I_R in the resistor necessary to produce any given phase angle θ between the total current and the applied voltage is $I_X \cot \theta$. Get an equation for $dI_R/d\theta$.

25. The power P in a certain ac circuit is to remain constant. Then as the impedance phase angle θ is varied, the apparent power will always be $P_{app} = P \sec \theta$. Find $dP_{app}/d\theta$.

26. The figure of merit Q of a coil is equal to the tangent of the impedance phase angle θ of the coil. Get an equation for the rate of change of Q with respect to θ.

27. A radar antenna rotating at 30 revolutions per minute is located aboard a ship which is 6 miles from a straight shore line. How fast does the radar beam travel along the shore line when the beam makes an angle of 50° with the shore?

11-11 Inverse trigonometric functions. Suppose we are given that

$$u = \sin y$$

This information can also be written in either of these ways:*

$$y = \sin^{-1} u \qquad \text{or} \qquad y = \arcsin u$$

which can be read aloud as (*a*) "y is equal to the arc sine of u," or (*b*) "y is the angle whose sine is u," or (*c*) "y is equal to the inverse sine of u." Similarly, if $y = \tan^{-1} u$, this is taken to mean that y is the angle whose tangent is u, and we say that "y is the arc tangent (or inverse tangent) of u." In cases like these examples y is called an *inverse trigonometric* (or inverse circular) *function.*

Figure 11-15 illustrates the relation between an ordinary or *direct*

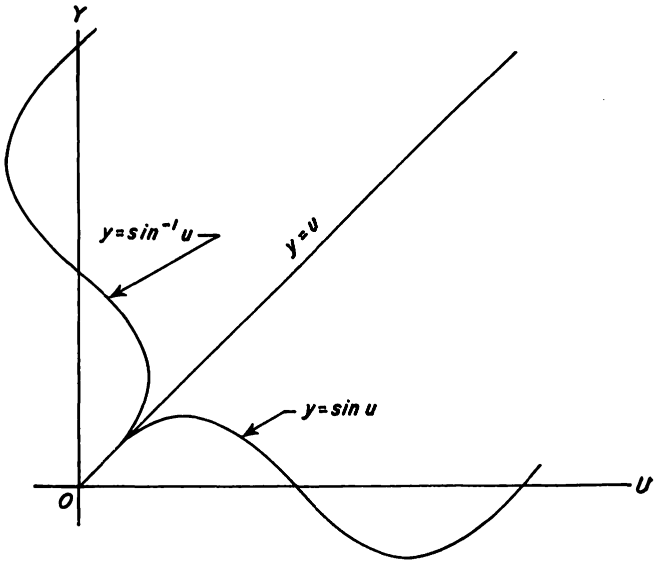


Fig. 11-15

trigonometric function and the corresponding inverse function. The horizontal graph displays $y = \sin u$, while the vertical graph presents the inverse function $y = \sin^{-1} u$. Note that the inverse sine curve appears as the image of the sine curve, in a plane mirror placed along the straight line $y = u$. Similarly, the curve $y = \tan^{-1} u$ would appear as a "reflection" in the line $y = u$ of the curve $y = \tan u$, etc.

It should be emphasized that writing $y = \tan^{-1} u$, for example, is precisely the same as writing $u = \tan y$.

Figure 11-16 shows the graphs of various inverse trigonometric functions.

* The form $(\sin u)^{-1}$ is used to indicate the reciprocal, or minus first power, of the sine of u. For other powers of the direct trigonometric functions, however, we use forms like $\sin^2 u$, etc.

The sine and cosine functions, it is recalled, may have values in the range from -1 to $+1$. Correspondingly, there exist values of $\cos^{-1} u$ and of $\sin^{-1} u$ for values of u in the range from -1 to $+1$. To indicate this fact we say that "the functions $\cos^{-1} u$ and $\sin^{-1} u$ are defined in the interval -1 to $+1$."

If $y = \tan^{-1} u$ or if $y = \cot^{-1} u$, u may have any value in the range $-\infty$ to $+\infty$. For both $y = \sec^{-1} u$ and $y = \csc^{-1} u$, permissible values of u are in the intervals $-\infty$ to -1 and $+1$ to $+\infty$.

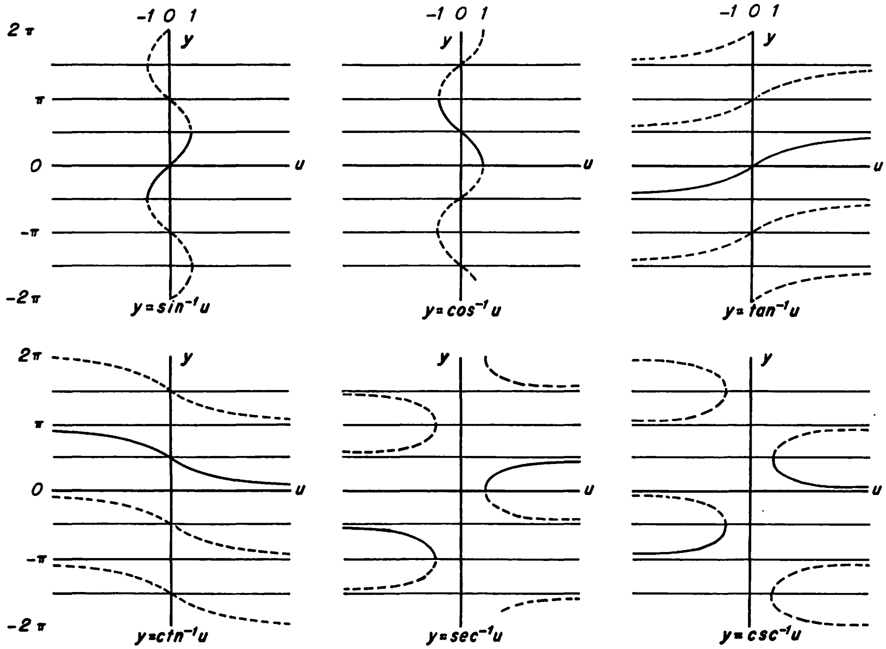


Fig. 11-16

11-12 Principal values of the inverse trigonometric functions. Each of the graphs of Fig. 11-16 can be extended indefinitely upward and downward, so that in general there exists an infinite number of values of y for each value of u. If there were no restrictions imposed, the result would be an infinite number of solutions for each equation like $y = \tan^{-1} u$, for example. Suppose, for instance, that $y = \tan^{-1} 1$. Then y can take an infinite number of values, each satisfying this equation. Among these values are $y = \pi/4$, $y = 5\pi/4$, $y = 9\pi/4$, etc. You should confirm this situation by referring to the figure.

However, it is the custom to restrict the solutions to such equations, *unless otherwise stated*, to those within arbitrary ranges called the *principal values* of the functions. These ranges are indicated by the *solid-line* graphs of Fig. 11-16. For convenience, these ranges of principal values

are listed in Table 11-1. As an example, the equation $y = \tan^{-1} 1$ is taken to mean that $y = \pi/4$, unless some indication is given that another value is intended.*

Table 11-1

Function, y..........	$\sin^{-1} u$	$\cos^{-1} u$	$\tan^{-1} u$	$\cot^{-1} u$	$\sec^{-1} u$	$\csc^{-1} u$
Range of principal values.............	$-\frac{\pi}{2}$ to $+\frac{\pi}{2}$	0 to $+\pi$	$-\frac{\pi}{2}$ to $+\frac{\pi}{2}$	0 to $+\pi$	$-\pi$ to $-\frac{\pi}{2}$ and 0 to $+\frac{\pi}{2}$	$-\pi$ to $-\frac{\pi}{2}$ and 0 to $+\frac{\pi}{2}$

11-13 Derivatives of the inverse trigonometric functions. Differentiation formulas for the inverse trigonometric functions are readily obtained. Let us first differentiate

$$y = \sin^{-1} u \tag{27}$$

This is equivalent to

$$u = \sin y$$

Differentiating with respect to x,

$$\frac{du}{dx} = \cos y \frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{1}{\cos y} \frac{du}{dx} \tag{28}$$

From trigonometry, $\cos y = (1 - \sin^2 y)^{1/2}$, which in the present case gives

$$\cos y = \sqrt{1 - u^2} \tag{29}$$

Substituting (27) and (29) in (28),

$$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx} \tag{30}$$

There remains only a question as to whether the positive or the negative square root should be selected in (30). Since $\sin^{-1} u$ has a positive slope throughout its range of principal values, as shown in Fig. 11-16, we use the positive root as indicated in (30).

If we let $y = \cos^{-1} u$ and proceed along lines parallel to the above, we get

$$\frac{d}{dx} \cos^{-1} u = -\frac{1}{\sqrt{1 - u^2}} \frac{du}{dx} \tag{31}$$

* The ranges of principal values of certain of the inverse trigonometric functions are given differently by some writers. In some books the symbols of the inverse trigonometric functions begin with capital letters (as, $\text{Tan}^{-1} u$ or $\text{Arctan}\ u$) when values within the ranges of principal values of the functions are intended.

Next we differentiate

$$y = \tan^{-1} u \tag{32}$$

or

$$u = \tan y$$

Differentiating with respect to x,

$$\frac{du}{dx} = \sec^2 y \frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \frac{du}{dx} \tag{33}$$

From trigonometry, $\sec^2 y = 1 + \tan^2 y$, or

$$\sec^2 y = 1 + u^2 \tag{34}$$

Substitution of (32) and (34) into (33) yields

$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1 + u^2} \frac{du}{dx} \tag{35}$$

If we let $y = \cot^{-1} u$, a similar procedure gives

$$\frac{d}{dx} \cot^{-1} u = -\frac{1}{1 + u^2} \frac{du}{dx} \tag{36}$$

We next differentiate

$$y = \sec^{-1} u \tag{37}$$

or

$$u = \sec y$$

Differentiation gives

$$\frac{du}{dx} = \sec y \tan y \frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \frac{du}{dx} \tag{38}$$

But $\tan y = (\sec^2 y - 1)^{1/2}$, or

$$\tan y = \sqrt{u^2 - 1} \tag{39}$$

Substituting (37) and (39) into (38),

$$\frac{d}{dx} \sec^{-1} u = \frac{1}{u\sqrt{u^2 - 1}} \frac{du}{dx} \tag{40}$$

If, on the other hand, we let $y = \csc^{-1} u$, a similar derivation yields

$$\frac{d}{dx} \csc^{-1} u = -\frac{1}{u\sqrt{u^2 - 1}} \frac{du}{dx} \tag{41}$$

If, in the foregoing differentiation formulas, we let the inverse functions take values outside their ranges of principal values, the signs of the derivatives may have to be reversed (except in the cases of $\tan^{-1} u$ and

$\cot^{-1} u$). Observing the slopes of the curves of Fig. 11-16 should help in visualizing this situation.

QUESTIONS

1. Read each of the following statements aloud in three different ways:

(a) $y = \sin^{-1} x$ (c) $y = \tan^{-1} \theta$ (e) $y = \sec^{-1} A$
(b) $y = \cos^{-1} u$ (d) $y = \cot^{-1} \phi$ (f) $y = \csc^{-1} B$

2. In each of the equations of question 1, for what values of the independent variable will the equation have meaning?

3. What purpose is served by our adoption of the idea of a *range of principal* values of the inverse trigonometric functions?

4. State the ranges of principal values of each of the six inverse trigonometric functions treated in the preceding sections.

5. State the equations used in differentiating the six inverse trigonometric functions treated in the preceding sections.

PROBLEMS

In Probs. 1 to 14 differentiate with respect to x.

1. $y = \sin^{-1} 2x$
2. $y = \cos^{-1} x^2$
3. $y = \tan^{-1} x^2$
4. $y = \cot^{-1} 2x^2$
5. $y = \cos^{-1} x/2$
6. $y = \sec^{-1} (x^2 + 1)^{1/2}$
7. $y = \csc^{-1} 3x^5$
8. $y = \sin^{-1} 4x^4$
9. $y = \sin^{-1} (1 - x^2)^{1/2}$
10. $y = \tan^{-1} (2x - 1)$
11. $y = \sin^{-1} x \sin x$
12. $y = x^2 \cos^{-1} x$
13. $y = \cot^{-1} (\csc x + \cot x)$
14. $y = \sec^{-1} (x^2 + 2x)$

15. Derive Formula (31).

16. Derive Formula (36).

17. Derive Formula (41).

18. The distance s meters from the end of a certain transmitting antenna at which the current is I amperes is given by $s = (\lambda/2\pi) \sin^{-1} (I/I_0)$, where λ is the operating wavelength and I_0 is the maximum value of the current along the antenna. Find ds/dI.

19. The angle of grid-current flow in a triode class C amplifier is $\theta_g = 2 \cos^{-1}(V_c/V_s)$, where V_c is the bias voltage and V_s the peak excitation voltage. Find $d\theta_g/dV_c$.

20. The impedance phase angle of an inductor is related to the Q of the inductor by the relation $\theta = \tan^{-1} Q$. If $Q = 100$, what would be the approximate change in θ resulting from an increase in Q to 101?

21. A simple low-pass four-terminal network uses a series resistor R followed by a shunt capacitor C. The phase shift introduced into a certain circuit by this network is $\phi = \tan^{-1} \omega RC$. Find a formula for the rate of change of ϕ with respect to C.

22. A plane flies at 200 miles per hour on a straight course past a transmitting station, passing the station at a distance of 6 miles. An automatic direction finder (ADF) aboard the plane continuously indicates the bearing θ of the transmitter relative to the course of the plane. How fast is the ADF reading changing, in degrees per second, 3 minutes after the plane passes the station?

23. A guy wire is attached to a ring 35 feet above ground on an antenna pole. If the length s of the guy wire is changed, the angle between the pole and the wire is also changed. Calling this angle ϕ, find $d\phi/ds$ when $s = 52$ feet.

24. Light emerges from a refracting element at an angle $r = \sin^{-1}[(\sin i)/\mu]$, where i is the angle of incidence of the light and μ is the index of refraction of the refracting material. If the design of the element is changed to use various materials, find $dr/d\mu$.

11-14 Integrals yielding trigonometric forms. The differentiation formulas which we have developed enable us to integrate several functions. To integrate $\sin u\, du$ we begin with (14):

$$\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$$

or

$$\sin u\, du = -d(\cos u)$$

Integrating, we get the formula

$$\int \sin u\, du = -\cos u + C \tag{42}$$

A somewhat similar treatment of (13) gives

$$\int \cos u\, du = \sin u + C \tag{43}$$

If we turn to (23), we have

$$\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$$

or

$$\sec^2 u\, du = d(\tan u)$$

so that

$$\int \sec^2 u\, du = \tan u + C \tag{44}$$

Similarly, (24) can be used to give

$$\int \csc^2 u\, du = -\cot u + C \tag{45}$$

We turn next to (25):

$$\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{d}$$

or

$$\sec u \tan u\, du = d(\sec u)$$

so that

$$\int \sec u \tan u\, du = \sec u + C \tag{46}$$

A parallel treatment of (26) gives

$$\int \csc u \cot u\, du = -\csc u + C \tag{47}$$

Further trigonometric integrals are presented later in this book.

Example 1. If the peak value of a sine wave of current is $I_{\max}$, find the average value I_{av} of the current, taken over an interval of $\frac{1}{2}$ cycle.

Figure 11-17 shows the wave under consideration. Its equation is $i = I_{max} \sin \omega t$. Our problem is to find the average height of the curve over the base interval from $\omega t = 0$ to $\omega t = \pi$. To do this we shall find by integration the area under the curve over this interval; then, dividing this area by the length of the base, we shall get the average height of the curve:

$$I_{av} = \frac{1}{\pi} I_{max} \int_0^{\pi} \sin \omega t \, d(\omega t) = -\frac{1}{\pi} I_{max} \cos \omega t \Bigg]_{\omega t = 0}^{\pi}$$

$$I_{av} = \frac{2}{\pi} I_{max} = 0.637 I_{max} \qquad (48)$$

The latter is the familiar form which is arrived at only approximately, in more elementary courses, by averaging a finite number of values of current. It will be found that if the current is averaged over a ¼-cycle interval, as from $\omega t = 0$ to $\omega t = \pi/2$, a similar result is obtained. This comes from the symmetry between

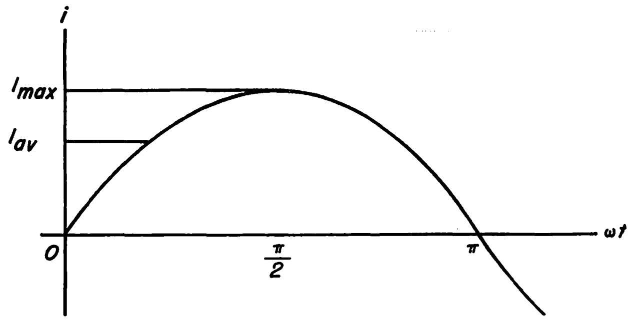


Fig. 11-17

succeeding ¼-cycles of the wave. On the other hand, if the average current is taken over a complete cycle (or a whole number of cycles) the average current is *zero* since for each alternation of current in a positive direction there is an equal alternation in the reverse direction. It is likewise found that a *cosine* function, or any other function whose graph has the form of a sine wave, regardless of its phase angle, has a definite integral equal to zero when the interval is a whole number of cycles. And therefore its average value is zero when taken over a whole number of cycles.

Example 2. Show that the average power P in an ac circuit is equal to $IV \cos \theta$, where I and V are the effective current and effective voltage, respectively, and θ is the phase angle between them.

The power at any instant is the product of the instantaneous current i and the instantaneous voltage v: $p = iv$. But the voltage wave is of the form $v = V_{max} \sin \omega t$, while the current wave has the form $i = I_{max} \sin (\omega t + \theta)$. Then

$$p = I_{max} V_{max} \sin \omega t \sin (\omega t + \theta)$$

A trigonometric identity shows that the product of the sines of two angles is

$$\sin \alpha \sin \beta = \tfrac{1}{2} [\cos (\alpha - \beta) - \cos (\alpha + \beta)]$$

Letting $\alpha = \omega t + \theta$ and $\beta = \omega t$,

$$p = \tfrac{1}{2} I_{\max} V_{\max} [\cos \theta - \cos (2\omega t + \theta)]$$

To get the average value of this power over ½ cycle we obtain by integration the area under the power curve for an interval of 0 to π radians; dividing the result by π gives the average height of the power curve:

$$P = \frac{I_{\max} V_{\max}}{2\pi} \int_0^{\pi} [\cos \theta - \cos (2\omega t + \theta)]\, d(\omega t)$$

Integrating term by term,

$$P = \tfrac{1}{2} I_{\max} V_{\max} \cos \theta$$

But it is well known (and will be demonstrated in Chap. 15) that the effective currents and voltages I and V are given respectively by $I_{\max}/\sqrt{2}$ and by $V_{\max}/\sqrt{2}$. Then

$$P = \frac{I_{\max}}{\sqrt{2}} \frac{V_{\max}}{\sqrt{2}} \cos \theta = IV \cos \theta \tag{49}$$

Example 3. Evaluate $\int \sin \phi \cos \phi \, d\phi$.

Let u represent $\sin \phi$. Then $du = \cos \phi \, d\phi$, so that the given integral is of the form $\int u \, du$. This has the value $u^2/2 + C$, or

$$\int \sin \phi \cos \phi \, d\phi = \tfrac{1}{2} \sin^2 \phi + C$$

Example 4. Evaluate $\int x \cos x^2 \, dx$.

Let $u = x^2$ and $du = 2x \, dx$. Rewriting the given integral

$$\int x \cos x^2 \, dx = \tfrac{1}{2} \int \cos x^2 (2x \, dx)$$

we see that it takes the form $(\int \cos u \, du)/2$, so that

$$\int x \cos x^2 \, dx = \tfrac{1}{2} \sin x^2 + C$$

Example 5. A voltage $v = V_{\max} \sin \omega t$ is applied across an inductor. Neglecting the resistance of the inductor, calculate the *opposition* offered by the inductor to the current flow.

We calculate first the current in the inductor. At any instant, this is $i = -(1/L)\int v \, dt$:

$$i = -\frac{1}{L} V_{\max} \int \sin \omega t \, dt = -\frac{V_{\max}}{\omega L} \int \sin \omega t \, d(\omega t) = \frac{V_{\max}}{\omega L} \cos \omega t$$

The current is seen to be a cosine wave whose peak value is $V_{\max}/\omega L$. Using the familiar fact (to be demonstrated in Chap. 15) that the effective value of a sine or cosine wave is equal to the peak value divided by $\sqrt{2}$, we write the effective voltage V and the effective current I in the inductor

$$V = \frac{V_{\max}}{\sqrt{2}} \qquad \text{and} \qquad I = \frac{I_{\max}}{\sqrt{2}} = \frac{V_{\max}}{\sqrt{2}\, \omega L}$$

Consider the *opposition* to the current flow as being equal to the effective voltage divided by the effective current:

$$\frac{V}{I} = \frac{V_{\max}/\sqrt{2}}{V_{\max}/\sqrt{2}\,\omega L} = \omega L$$

This opposition is referred to as the *inductive reactance* of the inductor and is measured in ohms:

$$X_L = \omega L \qquad \text{ohms} \tag{50}$$

11-15 Area in polar coordinates. Figure 11-18 shows a graph of a function $r = f(\theta)$; that is, the length r of the radius vector varies in a known manner as the angle θ (between the radius vector and the polar axis OX) is changed. Let it be desired to get a formula for the area A included within this polar graph between the positions $\theta = a$ and $\theta = b$ of the radius vector.

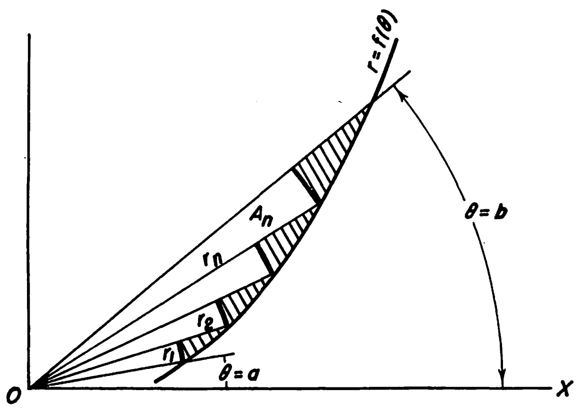


Fig. 11-18

To do this, consider the region from $\theta = a$ to $\theta = b$ to be broken up into a large number of small equal angles $\Delta\theta$, as indicated. This is accomplished by drawing successive radii r_1, r_2, etc., from the pole O.

Next, we draw small arcs of circles, as shown, each arc having O as its center. The radii of these arcs are r_1, r_2, etc.

The desired area A is seen to be *approximately* equal to the sum of a large number of small *circular sectors* ("pieces of pie"). The error in this approximation is indicated by the shaded portions of the figure. And we should expect the amount of this error to diminish if we further subdivided the sectors into smaller and smaller sectors.

The area A_n of any sector of Fig. 11-18 (say the nth sector), whose radius is r_n and whose central angle is $\Delta\theta$, will be simply its proportionate share of the area of a circle having the same radius. That is, $A_n = (\Delta\theta/2\pi)\pi r^2 = r^2\,\Delta\theta/2$.

Consider the desired area A as being equal to the limit approached by the sum of the sectors as their number n increases without bound and $\Delta\theta$ approaches zero:

$$A = \lim_{n \to \infty} [\tfrac{1}{2}(r_1^2\,\Delta\theta + r_2^2\,\Delta\theta + \cdots + r_n^2\,\Delta\theta)]$$

By the fundamental theorem [Eq. (19), Chap. 10], this is*

$$A = \tfrac{1}{2}\int_{\theta=a}^{b} r^2\,d\theta \qquad \textbf{(51)}$$

Example. Find the entire area contained in the curve $r = \cos 2\theta$ (Fig. 11-19).

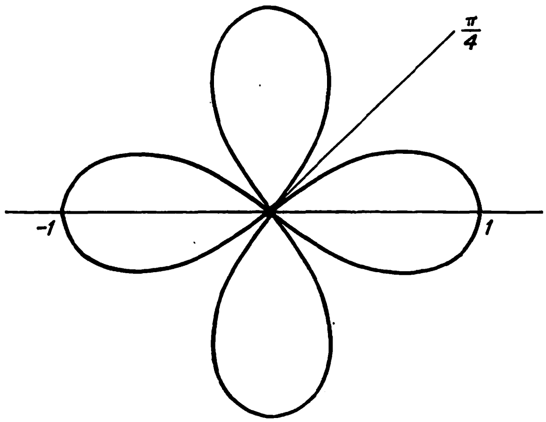


Fig. 11-19

The total area is eight times that between $\theta = 0$ and $\theta = \pi/4$. Applying (51) and identity 9 from Table 1, we get

$$A = 4\int_0^{\pi/4} \cos^2 2\theta\,d\theta = 2\int_0^{\pi/4} (1 + \cos 4\theta)\,d\theta = \pi/2 \text{ area units}$$

11-16 Integrals yielding inverse trigonometric forms. The first of these forms which we consider is

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C \qquad \textbf{(52)}$$

The correctness of this formula is shown by differentiating its right member according to (30):

$$\frac{d}{dx}\left(\sin^{-1}\frac{u}{a} + C\right) = \frac{1}{\sqrt{1 - u^2/a^2}}\,\frac{1}{a}\,\frac{du}{dx}$$

* Note that we do not depend upon (51) to obtain the formula for the area of a circle, which formula we used in deriving (51). The area of a circle can be calculated in rectangular coordinates in the manner of Sec. 16-2. In applying (51), it is advisable for simplicity and accuracy to find the smallest portion of the area from which the entire area can be deduced from considerations of symmetry.[1]

Multiplying both members by dx and clearing of fractions,

$$d\left(\sin^{-1}\frac{u}{a} + C\right) = \frac{du}{\sqrt{a^2 - u^2}} \tag{53}$$

The right member of (53) is identical with the integrand in (52); therefore (52) is correct.

Other integrals are

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \tag{54}$$

and

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C \tag{55}$$

which can be proved by differentiation.

Example 1. Evaluate $\int dx/(7 + 3x^2)$.

The given denominator can be written as the sum of two squares: $7 + 3x^2 = (\sqrt{7})^2 + (\sqrt{3}\,x)^2$. Letting $a = \sqrt{7}$, $u = \sqrt{3}\ x$, and $du = \sqrt{3}\ dx$, we apply (54):

$$\int \frac{du}{7 + 3x^2} = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\,dx}{(\sqrt{7})^2 + (\sqrt{3}x)^2} = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{7}}\tan^{-1}\frac{\sqrt{3}\,x}{\sqrt{7}} + C$$
$$= \frac{1}{\sqrt{21}}\tan^{-1}\left(\sqrt{\frac{3}{7}}\,x\right) + C$$

Example 2. Evaluate $\int dx/(x^2 + 4x + 7)$.

This is readily handled if we complete the square in the given denominator:

$$x^2 + 4x + 7 = x^2 + 4x + 4 + 3 = (x + 2)^2 + (\sqrt{3})^2$$

Let $u = x + 2$, $du = dx$, $a = \sqrt{3}$. Then by (54),

$$\int \frac{dx}{x^2 + 4x + 7} = \int \frac{dx}{(x + 2)^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{x + 2}{\sqrt{3}} + C$$

Example 3. Evaluate $\int dx/\sqrt{16 + 24x - 16x^2}$.

Completing the square in the quantity under the radical, we can express that quantity as the difference of two squares:

$$16 + 24x - 16x^2 = 25 - (9 - 24x + 16x^2) = 5^2 - (4x - 3)^2$$

Let $a = 5$, $u = 4x - 3$, $du = 4dx$. Then use (52):

$$\int \frac{dx}{\sqrt{16 + 24x - 16x^2}} = \frac{1}{4}\int \frac{4dx}{\sqrt{5^2 - (4x - 3)^2}} = \frac{1}{4}\sin^{-1}\frac{4x - 3}{5} + C$$

QUESTIONS

1. State the formulas which give the following integrals:

(a) $\int \sin u \, du$ (c) $\int \sec^2 u \, du$ (e) $\int \sec u \tan u \, du$
(b) $\int \cos u \, du$ (d) $\int \csc^2 u \, du$ (f) $\int \csc u \cot u \, du$

2. Give a formula for the area enclosed within a figure in polar coordinates.
3. State the formulas which give the following integrals:

(a) $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}}$ (b) $\displaystyle\int \frac{du}{\sqrt{a^2 + u^2}}$ (c) $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}}$

PROBLEMS

1-15 In Probs. 1 to 25 find y. Check by differentiation.

1. $y = \int \sin 2x \, dx$
2. $y = \int \sec^2 5\theta \, d\theta$
3. $y = \int \frac{1}{2} \cos \frac{1}{2} x \, dx$
4. $y = \int \csc^2 v \, dv$
5. $y = \int \sin 5x \cos 5x \, dx$
6. $y = \int \sin^2 x \cos x \, dx$
7. $y = \int \cos^3 \theta \sin \theta \, d\theta$
8. $y = \int \sin^{1/2} \theta \cos \theta \, d\theta$
9. $y = \int \sec^2 u \tan u \, du$ [HINT: Rewrite in form $\int \sec u (\sec u \tan u \, du)$.]
10. $y = \int \csc^3 3x \cot 3x \, dx$
11. $dy = \csc^2 u \cot u \, du$
12. $dy = \theta \sin \theta^2 \, d\theta$
13. $dy = x^2 \cos x^3 \, dx$
14. $dy = \theta \sin 4\theta^2 \, d\theta$
15. $dy = \phi \cos 3\phi^2 \, d\phi$

16-25

16. $dy = \dfrac{dz}{\sqrt{16 - z^2}}$
17. $dy = \dfrac{dt}{100 + t^2}$
18. $dy = \dfrac{dx}{x\sqrt{x^2 - 4}}$
19. $dy = \dfrac{dx}{1 + 9x^2}$
20. $dy = \dfrac{dx}{\sqrt{9 - 25x^2}}$
21. $dy = \dfrac{dx}{x^2 + 2x + 3}$
22. $dy = \dfrac{dx}{x\sqrt{4x^2 - 49}}$
23. $dy = \dfrac{dx}{\sqrt{1 + 4x - x^2}}$
24. $dy = \dfrac{dx}{\sqrt{21 + 12x - 9x^2}}$
25. $dy = \dfrac{dx}{(x + 1)\sqrt{2x + x^2}}$

26. Solve Example 3 of Sec. 11-14, using this time $u = \cos \phi$. Compare with answer shown in example. If the results differ, explain (Sec. 9-9).

27. The voltage applied to an inductor of 1 henry inductance and negligible resistance is $v_L = 220 \sin 55t$. Find a formula for the current through the coil. Show that this circuit operates in accordance with the principle, from elementary electricity, that the alternating current through an inductor of negligible resistance lags the applied voltage by 90°.

28. The current supplied to an initially discharged 8-microfarad capacitor was $i_C = 0.001 \sec^2 (t/2)$. Find the capacitor voltage after $\pi/2$ seconds.

29. It is desired to obtain from a transformer a secondary emf $v_2 = 21 \cos 3{,}213t$. If the mutual inductance between the windings is 20 henrys, find the primary current waveform required.

30. Same as Prob. 29, except let $v_2 = 1/(4t^2 + 9)$.

31. Find the area described by the radius vector of the curve $r = 2 \sec 2\theta$ as it rotates between the positions $\theta = 0$ and $\theta = \pi/8$.

32. When a semiautomatic key is adjusted to transmit International Morse code at 35 words per minute, the dot contact executes approximately simple harmonic motion with a speed $\mathbf{v} = 14\pi \cos 28\pi t$ millimeters per second. Find the total distance traveled by the contact in making five dots. (Consider all motion in this case as positive.)

33. A microphone has a cardioid directional pattern given approximately by $r = k(1 - \cos \theta)$. Find the rms value of this pattern (that is, find the radius of a circle having the same area). [HINT: In completing the integration, use the identity $\cos^2 \theta = (1 + \cos 2\theta)/2$.]

REFERENCE

1. C. R. WYLIE: "Calculus," pp. 270–274; McGraw-Hill Book Company, Inc., New York, 1953.

12

Logarithmic and Exponential Functions

The functions which we consider next are *logarithmic* functions, having the form $y = \log u$. We shall also take up *exponential* functions, those whose form is $y = a^u$. The logarithmic and the exponential functions are of great importance in the study of electricity.

12-1 Logarithmic functions. Consider the function

$$y = \log_a u \tag{1}$$

(read "y equals the logarithm to the base a of u.") Here we have a quantity y which varies according to the logarithm of an independent variable u. We have not yet specified the value of the base a of the system of logarithms being used. Two systems of logarithms are in common use: those to the base 10 and those to the base e ($= 2.718 \cdots$).

Let us see what a graph of (1) would look like. For an illustration, let $a = 10$, so that $y = \log_{10} u$. Then we may select a few values from a table of common logarithms, as indicated in Table 12-1. These points

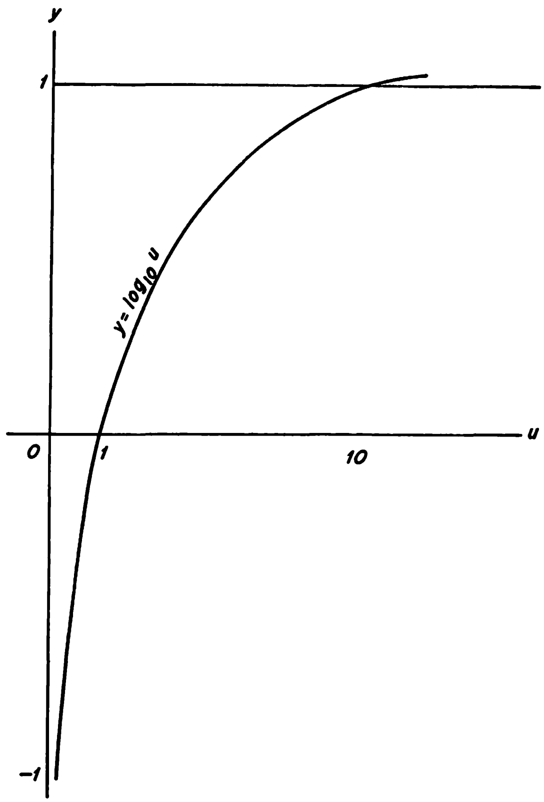


Fig. 12-1

are plotted in Fig. 12-1, giving a graph of the function $y = \log_{10} u$. The form of the graph will be much the same whenever the base is a positive number greater than 1. The form of the graph should be noted carefully, observing that

1. When $u = 1$, then $\log u = 0$.
2. As u grows smaller, approaching zero, then $\log u$ increases negatively without bound.

Table 12-1

u	0.01	0.1	1	2	4	6	8	10	100
$y = \log_{10} u$	$8.0000 - 10 (= -2)$	$9.0000 - 10 (= -1)$	0.0000	0.3010	0.6021	0.7782	0.9031	1.0000	2.0000

3. The graph does not exist for negative values of u; that is, a negative number has no real logarithm to a real base.
4. When u equals the base a (in this case, 10), then $\log u = 1$.

Although it is assumed that you have a working knowledge of the use of logarithms, it is well to review here the definition of a logarithm:

The logarithm of a number u to a base a is defined as the exponent of the power to which the base must be raised to equal the number u.

Thus, if $u = a^y$, then by this definition, $y = \log_a u$.

Many times confusion regarding logarithms is removed simply by recalling just what logarithms are.

Our first task here will be to find the *slope* of the graph of Fig. 12-1 at any point; that is, we shall find the derivative of $y = \log_a u$.

12-2 Derivative of a logarithmic function. To get a differentiation formula for a new type of function we customarily use the delta method. Here, we let

$$y = \log_a u \tag{1}$$

If we let u go through a change Δu, this equation becomes

$$y + \Delta y = \log_a (u + \Delta u) \tag{2}$$

Subtracting (1) from (2),

$$\Delta y = \log_a (u + \Delta u) - \log_a u$$

or

$$\Delta y = \log_a \frac{u + \Delta u}{u} \tag{3}$$

so that

$$\frac{\Delta y}{\Delta u} = \frac{1}{\Delta u} \log_a \frac{u + \Delta u}{u} \tag{4}$$

[Here, we might at first think that the required derivative formula could be had simply by letting Δu approach zero. But such a step would make the logarithm in the right member approach $\log_a (u/u) = \log_a 1 = 0$; consequently, the limit approached by the entire right member would be of the form 0/0—a meaningless expression. We find it desirable to change the form of (4) before taking limits.]

We introduce a new variable z, which we shall use to represent the quantity $u/\Delta u$. That is,

$$z = \frac{u}{\Delta u} \qquad \text{and} \qquad \frac{1}{\Delta u} = \frac{z}{u} \tag{5}$$

From (5), we observe that, as Δu approaches zero, z increases without bound. Substituting (5) in (4),

$$\frac{\Delta y}{\Delta u} = \frac{z}{u} \log_a \left(1 + \frac{1}{z}\right)$$

or

$$\frac{\Delta y}{\Delta u} = \frac{1}{u} \log_a \left(1 + \frac{1}{z}\right)^z \tag{6}$$

We are now ready to let Δu approach zero as a limit. As we have seen, this makes z increase without bound. Then, interestingly enough, the

limit approached by $(1 + 1/z)^z$ becomes identically equal to e (Sec. 4-8). The result is

$$\frac{dy}{du} = \frac{1}{u} \log_a e \tag{7}$$

or

$$\frac{d}{du} \log_a u = \frac{1}{u} \log_a e \tag{8}$$

This may be written in the more general form

$$\frac{d}{dx} \log_a u = \frac{1}{u}\frac{du}{dx} \log_a e \tag{9}$$

which provides for cases where u is a function of some other variable, x. Putting this rule (9) into words,

The derivative of the logarithm to the base a of a function u is equal to the reciprocal of the function times the derivative of the function times the logarithm to the base a of e.

If we differentiate the logarithm *to the base* e of a function u, (9) will take as its right member $(1/u)(du/dx) \log_e e$, but we note from the definition of a logarithm that $\log_e e = 1$, so that

$$\frac{d}{dx} \log_e u = \frac{1}{u}\frac{du}{dx} \tag{10}$$

In this book, we use $\ln u$ to indicate $\log_e u$. Then (10) becomes

$$\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx} \tag{11}$$

The simplicity of (11) as compared with (9) is a reason for preferring that logarithms be expressed to the base e, rather than, say, 10, when they are to be differentiated. Logarithms to the base 10 are preferred for *numerical calculations*, but we prefer to use the base e in expressing functions when we intend to perform *calculus operations* upon them.*

Example 1. The insulation resistance of a shielded cable is given by $R = (\rho/2\pi) \ln (r_2/r_1)$ ohms per meter, where ρ is the resistivity of the insulation material and r_1 and r_2 are, respectively, the inner and the outer radii of the insulation. Find the rate of change of R with respect to r_2.

This rate is obtained by (11):

$$\frac{dR}{dr_2} = \frac{\rho}{2\pi}\frac{r_1}{r_2}\frac{1}{r_1} = \frac{\rho}{2\pi r_2} \qquad \text{ohms per meter per meter}$$

* Logarithms to the base e (called *natural* or *napierian* logarithms) are sometimes indicated by $\log_\epsilon u$, or simply $\log u$. The latter notation is sometimes confusing, since it may also be used to indicate the logarithm to some other base, such as 10. But as you become familiar with the subject matter of a problem, there is seldom any confusion about which base is intended.

Example 2. Prove the formula

$$\int \sec u\, du = \ln(\sec u + \tan u) + C \tag{12}$$

Differentiating the right member,

$$\frac{d}{du}\ln(\sec u + \tan u) = \frac{1}{\sec u + \tan u}(\sec u \tan u + \sec^2 u)$$

or

$$d[\ln(\sec u + \tan u)] = \sec u\, du$$

Since this has as its right member the integrand in the given formula, that formula must be correct. A similar formula

$$\int \csc u\, du = \ln(\csc u - \cot u) + C \tag{13}$$

is left to you for proof (Prob. 11).

Sometimes we wish to differentiate a logarithmic function when it is expressed to the base 10. Equation (9) gives us

$$\frac{d}{dx}\log_{10} u = \frac{1}{u}\frac{du}{dx}\log_{10} e \tag{14}$$

The quantity $\log_{10} e$ appearing in (14) is often needed. For brevity it is called M. From tables of logarithms to the base 10, it is found to have the value $M = 0.4343$. Then

$$\frac{d}{dx}\log_{10} u = \frac{M}{u}\frac{du}{dx} = \frac{0.4343}{u}\frac{du}{dx} \tag{15}$$

Example 3. The inductance of a parallel-wire transmission line is given by

$$L = 0.281 \log_{10}\frac{b}{a} + 0.03 \qquad \text{microhenrys per foot}$$

where a is the radius of the wires and b is the spacing between their centers. Find the rate of change of inductance as the spacing of the wires is varied. (Let a and b be measured in inches.)

From (15),

$$\frac{dL}{db} = 0.281\frac{0.4343}{b/a}\frac{1}{a} = \frac{0.122}{b} \qquad \text{microhenrys per foot per inch}$$

PROBLEMS

In Probs. 1 to 10 find dy/dx.

1. $y = \ln 2x$
2. $y = \ln\dfrac{1}{1-x}$
3. $y = 2\ln x$
4. $y = \ln x^2$
5. $y = \dfrac{x^2}{\ln(x-1)}$
6. $y = \log_{10}(x^2 + x)$
7. $y = \log_{10}\dfrac{x^2}{M}$
8. $y = 3\log_{10} x$
9. $y = 5\log_{10}\dfrac{x}{x^2+1}$
10. $y = \log_{10}\sin x$

11. Prove the integration formula $\int \csc u\, du = \ln(\csc u - \cot u) + C$.

12. In Example 1, above, find the rate of change of the insulation resistance of the cable as the inner radius is varied, keeping the outer radius constant.

13. In Example 3, above, find the rate of change of L with respect to a.

14. The inductance of a coaxial cable is given by $L = 0.140 \log_{10} (b/a) + 0.015$ microhenrys per foot, where a and b are radii of the inner and outer conductors, respectively. Find dL/da.

15. In the formula of Prob. 14, assume $a = \frac{3}{8}$ inch and $b = \frac{7}{8}$ inch. Find the approximate change in L resulting from an increase in b of 0.01 inch.

16. In the water-cooling system for a large transmitter the following relation gives the difference T_D in temperature between the inner and the outer walls of a pipe: $T_D = -(q/2\pi k) \ln (r_2/r_1)$, where r_1 and r_2 are the inner and the outer radii, respectively, of the pipe. For given values of q and k find the rate of change of T_D with respect to the inner radius.

17. When a metal is dipped into a solution containing ions of that metal, an emf is produced between the metal and the solution. Its value is $V = -(KT/nF) \ln (P/p)$. If K, T, n, F, and P are constants, find dV/dp.

18. The capacitance of a two-wire transmission line is $C = 3.68/\log_{10} (b/a)$ micromicrofarads per foot, where a is the radius of the wires and b is the spacing between their centers. Find dC/da, assuming b constant.

19. An air-dielectric capacitor is made up of two concentric cylinders of length l. Its capacitance is $C = 24.161/\log_{10} (r_2/r_1)$, where r_1 and r_2 are the radii of the cylinders (all dimensions are in meters). If l and r_1 are constants, find dC/dr_2.

20. The amount of preemphasis D in decibels effected by a certain network in the sound system of a television station varies with frequency f as $D = 10 \log_{10} (1 + 39.45f^2k^2)$, where k is a constant. Find a formula for dD/df.

12-3 Natural logarithms. Table 3, in the Appendix, gives values of natural logarithms of numbers (base e). In contrast to the case of logarithms to the base 10, the mantissa of the natural logarithm of a number is *not* the same as the mantissa of the logarithm of the same number multiplied by a power of 10. When a logarithm is required for a number which is outside the range of the tables, proceed as follows:

1. Multiply (or divide) the number repeatedly by 10 until the result falls within the range of the table.
2. Find the logarithm of this product.
3. Subtract ln 10 (= 2.3026) for each time you multiplied by 10 (or add ln 10 for each time you divided by 10) in step 1. The result is the desired logarithm.

To find the number corresponding to a given logarithm which falls outside the table:

1. Subtract ln 10 repeatedly from the given logarithm (or add ln 10 repeatedly) until the logarithm falls within the table.
2. Find the antilogarithm of the difference (or sum) found in step 1.
3. Multiply this antilogarithm by 10 for each time you subtracted ln 10 (or divide the antilogarithm by 10 for each time you added ln 10) in step 1.

If the natural logarithm of a number is multiplied by $\log_{10} e$ ($= M = 0.4343$), the result will be the logarithm of the same number to the base 10. If the logarithm to the base 10 is multiplied by $\ln 10$ ($= 1/M = 2.3026$), the result of the multiplication will be the natural logarithm of the number.

PROBLEMS

In Probs. 1 to 10 find the natural logarithms of the given numbers, using the tables.

1. 2.52
2. 1.97
3. 4.023
4. 5.444
5. 31.1
6. 26.8
7. 82.77
8. 1,824
9. 0.88
10. 0.00636

In Probs. 11 to 20 find in the tables the numbers having the given natural logarithms.

11. 1.5173
12. 2.2006
13. 3.2006
14. 2.0888
15. 0.9012
16. 4.3241
17. 6.5034
18. 9.0961 − 10
19. 4.7088 − 10
20. 6.6682 − 10

12-4 Logarithmic differentiation. Equation (11) is a powerful tool for the differentiation of a wide range of functions which are *not* initially logarithmic functions by a process called *logarithmic differentiation.* The steps are as follows:

1. Assign a symbol (say y) to the given function, setting up a formula equating y to the given function.
2. Take the logarithm of each side of this equation.
3. Differentiate both sides of the resulting equation.
4. Solve for the desired derivative.

Example 1. Differentiate

$$\sqrt{\frac{x^2+2}{x^2-2}}$$

We write

$$y = \sqrt{\frac{x^2+2}{x^2-2}}$$

$$\ln y = \tfrac{1}{2} \ln (x^2+2) - \tfrac{1}{2} \ln (x^2-2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+2} - \frac{x}{x^2-2} = -\frac{4x}{(x^2+2)(x^2-2)}$$

Multiplying this equation by the above equation for y,

$$\frac{dy}{dx} = -\frac{4x}{(x^2+2)^{1/2}(x^2-2)^{3/2}}$$

Example 2. Differentiate x^{x^2}.

We let

$$y = x^{x^2}$$

Then

$$\ln y = x^2 \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = x^2 \frac{1}{x} + (\ln x)2x = x(1 + 2 \ln x)$$

$$\frac{dy}{dx} = x^{x^2+1}(1 + 2 \ln x)$$

Example 3. Prove Formula (3) of Sec. 5-5:

$$\frac{d}{dx} x^n = nx^{n-1}$$

Letting $y = x^n$, we have

$$\ln y = n \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x} \qquad \text{or} \qquad \frac{d}{dx} x^n = nx^{n-1}$$

QUESTIONS

1. State the formula for the derivative of $\ln u$.
2. State the formula for the derivative of $\log_{10} u$.
3. Explain an advantage in using the base e, rather than some other base, in expressing a logarithmic function which is to be differentiated.
4. Explain in what way logarithms to the base e are less convenient for numerical calculations than logarithms to the base 10.
5. State the procedure for differentiating a function logarithmically.

PROBLEMS

Solve the following problems through logarithmic differentiation. In Probs. 1 to 10 obtain dy/dx.

1. $y = x^{3x-1}$
2. $y = (x + 1)^{1/2}/(x - 1)^{3/2}$
3. $y = e^{2x}$
4. $y = (x - 1)^{1/2}/(x - 2)^{1/2}$
5. $y = 4e^{4x}$
6. $y = x^{\cos x}$
7. $y = x^{\sqrt{x}}$
8. $y = (x + 1)^3/(x - 2)^2$
9. $y = \sin^x x$
10. $y = x^{2/\ln x}$

11. Over a certain interval the charge delivered to a capacitor varied thus: $q = 2t^{\sin t}$. Find an equation for the current flowing into the capacitor.

12. The current in an inductor had values during a certain interval equal to 2^{t^3} amperes. Find an equation for the induced emf if $L = 2$ henrys.

13. The primary current in a transformer varied thus: $i_1 = 0.1e^{-t^2}$ amperes. Find an equation for the induced secondary emf v_2 if $M = 0.8$ henry.

14. The charge delivered to a 10-microfarad capacitor varied as $q = \cos^t t$. Find the rate at which the capacitor voltage changed.

15. The current in a transformer primary varied according to the formula $i_1 = 2.2e^{-3t^3}$ amperes. If the mutual inductance between the primary and secondary windings was 2 henrys, find the induced secondary emf equation.

16. Using logarithmic differentiation, derive (*a*) the formula for the derivative of a product and (*b*) the formula for the derivative of a quotient.

17. Derive a formula for the derivative of u^v, where u and v are functions of x.

12-5 Exponential functions. One of the most useful kinds of functions is that group known as the *exponential* functions. An exponential function has the form

$$y = a^u \tag{16}$$

where a is a constant and u is a variable. An example is the function $y = 2^x$, whose graph is shown by the *solid* curve of Fig. 12-2.

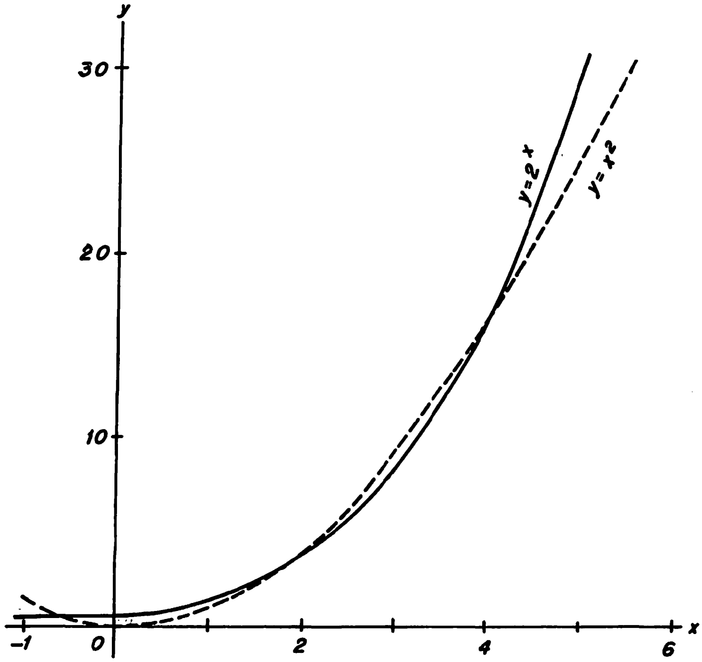


Fig. 12-2

The exponential functions must never be confused with *power* functions. The latter kind of function involves a variable quantity raised to a constant power (for example, $y = x^2$, whose graph is the *broken* line in Fig. 12-2).

Turning to Fig. 12-3, we find a graph of an exponential function, $y = a^u$, shown by the *solid* line. In this example, a is taken as being greater than one, while the variable u appears only in its *first power* in the exponent. We observe that

1. When $u = 0$, then $y = 1$.
2. If u is made more and more negative, then y approaches zero.
3. If u is made equal to $+1$, then y becomes equal to a.
4. If u becomes more and more positive, then y increases without bound.

The *broken* line graph of Fig. 12-3 shows the function $y = a^{-u}$. Here we note that

1. When $u = 0$, then $y = 1$.
2. If u becomes more and more negative, then y increases without bound.
3. If u is made equal to -1, then y becomes equal to a.
4. If u increases without bound, then y approaches zero.

The general forms of the graphs of Fig. 12-3 should be carefully noted.

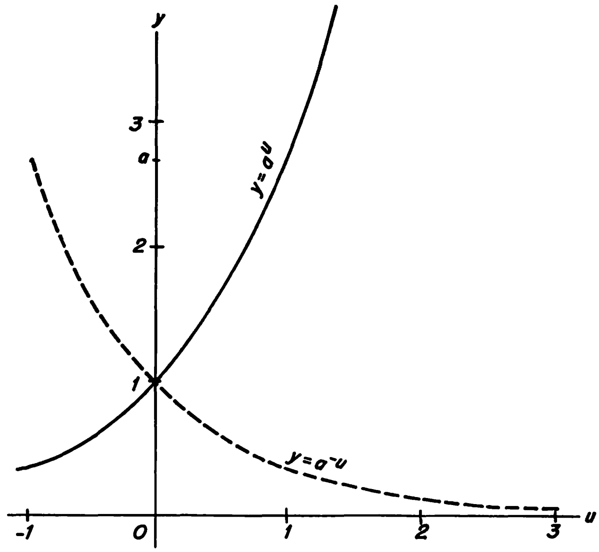


Fig. 12-3

12-6 Derivative of an exponential function. A formula for differentiating exponential functions is readily obtained. Taking logarithms in (16),

$$\ln y = u \ln a$$

which differentiates to give

$$\frac{1}{y}\frac{dy}{dx} = \frac{du}{dx} \ln a$$

Multiplying this equation by (16),

$$\frac{dy}{dx} = a^u \frac{du}{dx} \ln a$$

or

$$\frac{d}{dx} a^u = a^u \frac{du}{dx} \ln a \tag{17}$$

If the constant a in (17) should have the value e, this formula becomes even simpler, because $\ln e = 1$:

$$\frac{d}{dx} e^u = e^u \frac{du}{dx} \tag{18}$$

That is, the derivative of an exponential function (where e is the constant base) is equal to the given exponential function times the derivative of the exponent.

Example 1. Over a certain interval of its build-up time, the voltage delivered by a shunt generator varied according to $v = 20e^{0.1t}$. Find the rate of change of v with respect to t.

By (18), this is

$$\frac{dv}{dt} = 20e^{0.1t}(0.1) = 2e^{0.1t}$$

Example 2. In the study of probability and statistics, an important function has the form $y = ke^{-x^2}$, where k is a constant. (This function has a graph called the *normal probability curve.*) Find the rate of change of y with respect to x.

By (18),

$$\frac{dy}{dx} = -2kxe^{-x^2}$$

Sometimes the exponent in an exponential function is complicated, so that it is inconvenient to write the quantity in the form e^u. In such cases, we may use the *exponential operator*, exp. For example, the quantity

$$e^{\sqrt[3]{t^2-1}}$$

is more easily and clearly written

$$\exp \sqrt[3]{t^2 - 1}$$

QUESTIONS

1. Make a freehand graph of each of the following functions (assuming a to be greater than 1): (*a*) $y = \log_a u$ (*b*) $y = a^u$ (*c*) $y = a^{-u}$
2. State the formula for the differentiation of an exponential function, $y = a^u$; for the differentiation of the exponential function $y = e^u$.
3. What is the meaning of the operator exp?

PROBLEMS

In Probs. 1 to 10 differentiate with respect to x.

1. $y = e^{5x}$
2. $y = 12e^{12x}$
3. $y = e^{x-x^2}$
4. $y = e^{\sin x}$
5. $y = e^{2x} \sin 2x$
6. $y = x^2e^{-2x^2}$
7. $y = \sin^{-1} e^{2x}$
8. $y = e^{\ln x}$
9. $y = \tan e^{3x^2}$
10. $y = 10^{\cos x}$

11. A capacitor is charged to a voltage V by a battery connected to the capacitor through a resistor. The charging current may be shown to be $i = (V/R)e^{-t/RC}$. Find the rate of change of current at any instant.

12. A current $i_1 = 2.8e^{-0.01t}$ flows in the primary winding of a certain transformer. If the mutual inductance between the primary and secondary windings is 0.5 henry, find the induced secondary voltage at any time t.

13. A capacitance C is charged through a resistance R by a source having a constant voltage V. It can be shown that the capacitor voltage varies according to $v_C = V(1 - e^{-t/RC})$. Find dv_C/dt.

14. The power gain of a traveling-wave tube of length N cycles is $G = \frac{1}{9} \exp (2\pi \sqrt{3}\, CN)$. If C is a constant, find the rate at which G changes as N is varied.

15. In an electron tube, the maximum value of plate current available per unit area of filament, under temperature-limited conditions, is approximately $i = AT^2e^{-B/T}$, where T is the absolute temperature of the filament and A and B are constants. Find the rate of change of i with respect to T.

16. When the power supply to a motor is cut off at time $t = 0$, the speed ω decreases as follows: $\omega = \omega_0 e^{-0.15t}$, where ω_0 is the speed when $t = 0$. Find the rate at which the speed changes.

17. When a steady voltage V is applied to an inductance L in series with a resistance R, the resulting current is $i = (V/R)(1 - e^{-Rt/L})$. Find the resulting voltage v_L across the inductance.

18. The current needed to stimulate a nerve of length s is $ae^s(e^s - 1)$, where a is a constant. Find di/ds.

19. A low-frequency radio wave had a field intensity given by

$$\mathbf{E} = K \frac{\sqrt{P}}{D} \exp \frac{-K_1 D}{\lambda^{K_2}}$$

where D was the distance of transmission, P the antenna power, λ the wavelength, and K, K_1, and K_2 were constants. For a given power and wavelength, find $d\mathbf{E}/dD$.

20. In studying radiation from a heated object, such as the plate of a transmitting tube, the following formula was used:

$$J_\lambda = \frac{c_1}{\lambda^5[\exp (c_2/\lambda T) - 1]}$$

If c_1, c_2, and T are constants, find $dJ_\lambda/d\lambda$.

12-7 Inverse relation between exponential and logarithmic functions.

As we have seen, the two functions

$$y = e^u \qquad \text{and} \qquad y = \ln u$$

are in an inverse relation to each other. This relation is similar to that existing, for instance, between the functions

$$y = \sin x \qquad \text{and} \qquad y = \sin^{-1} x$$

discussed in the preceding chapter. In Fig. 12-4, we have graphed the function $y = \ln u$, shown in the *broken* line. There also appears in this diagram a *solid-line* graph, displaying the function $y = e^u$. (Here u is taken positive.) Note that these two graphs have the *mirror-reflection* relation to each other with respect to the line $y = u$, which we also found in the case of the trigonometric and the inverse trigonometric functions.

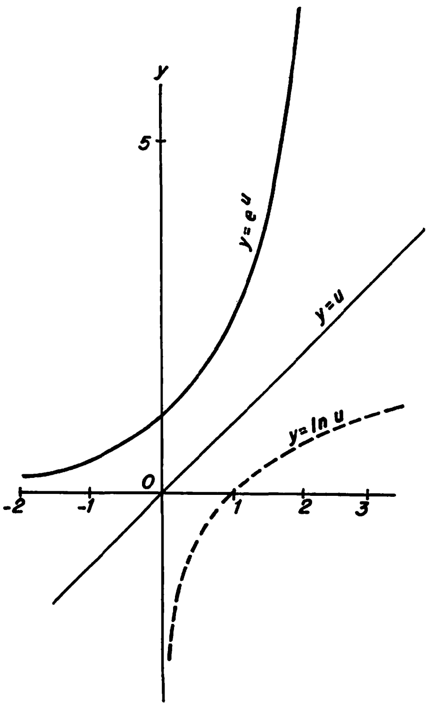


Fig. 12-4

Example 1. Given $y = e^{2x}$, find $\ln y$.
The logarithm of the right member is the exponent of the power to which the base e was raised to give the right member. Taking logarithms of both members then gives $\ln y = 2x$.

Example 2. Given $y = \ln 3x$, find x.
Taking antilogarithms of both sides of this equation,

$$e^y = 3x \qquad \text{or} \qquad x = \tfrac{1}{3}e^y$$

Example 3. Simplify $y = e^{\ln 2x}$.
Taking logarithms, we find $\ln y = \ln 2x$. Since the *logarithms* of y and of $2x$ are equal, we see that $y = 2x$.

We have noted the simplicity of the derivative of an exponential function when the exponential base is e. In practice, if we are given an exponential function using some base a other than e, we nearly always express the function in terms of e before differentiating it. We accomplish this by the formula

$$a^u = e^{u \ln a} \tag{19}$$

which is readily proved by taking logarithms of each member.

Example 4. Given $y = 10^x$, find dy/dx.
This problem can be solved in either of two ways: (a) Taking logarithms in the given equation, and differentiating:

$$\ln y = x \ln 10 = \frac{x}{M}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{M} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{M}10^x$$

or (*b*) first expressing the given function in terms of e by (19), and then differentiating:

$$y = 10^x = e^{x \ln 10} = e^{x/M}$$
$$\frac{dy}{dx} = \frac{1}{M} e^{x/M}$$

These two forms of the solution are seen to be equivalent, by (19).

12-8 Tables of the exponential function. A brief table of values of the exponential function e^x is included in the Appendix (Table 4). More complete tables are found in books of mathematical tables. The use of such tables is made clear by means of an example. To keep your orientation in solving problems, you should keep constantly in mind that *an exponential expression is an antilogarithm.*

Example 1. Find the numerical value of $y = 6.5e^{0.05}$.
From Table 4, we get $e^{0.05} = 1.051$. Multiplying, we have $y = 6.5(1.051) = 6.832$.

The problem may also be solved by means of the logarithms to the base 10 of the exponential quantities which are included in Table 4. (The base 10 is used here for convenience in numerical calculation.) The above example is reworked below by this method.

Example 2. Find the numerical value of $y = 6.5e^{0.05}$.
From Table 2 (Common Logarithms) we get $\log_{10} 6.5 = 0.8129$. In Table 4, we find $\log_{10} e^{0.05} = 0.0217$. Adding these two logarithms, we have $\log_{10} 6.5e^{0.05} = 0.8346$. From Table 2, the antilogarithm of this quantity is 6.832. (If the exponent is negative, we may use the column headed e^{-x} when using the method of Example 1, above. When using the method of Example 2, a quantity e^{-x} requires that we *subtract,* rather than add, the logarithm to the base 10 of e^x.)

Such problems can be worked by means of the table of natural logarithms (Table 3) if desired, as when the desired values are not given in the table of exponentials. The above problem is reworked below by this method.

Example 3. Find the numerical value of $y = 6.5e^{0.05}$.
Taking logarithms in the given equation,

$$\ln y = \ln 6.5 + 0.05 = 1.9218$$

Looking up the antilogarithm of 1.9218 in Table 3, we get 6.832.

(In some cases, the accuracy of results varies slightly, depending upon the choice among these tables and methods.)

Sometimes we have *to find the value of an unknown exponent.* This may be done as shown in Example 4.

Example 4. The discharge current of a capacitor was $i = 0.01e^{-100t}$. When was $i = 0.002$ ampere?

Substituting $i = 0.002$, and rearranging,

$$e^{-100t} = \frac{0.002}{0.01} = \frac{1}{5} \qquad \text{or} \qquad e^{100t} = 5$$

Taking logarithms,

$$100t = \ln 5 = 1.6094$$
$$t = 0.0161 \text{ second}$$

PROBLEMS

In Probs. 1 to 10 find the numerical values of the given expressions by any method.

1. e^2
2. $e^{0.1}$
3. $e^{0.5}$
4. $e^{-0.2}$
5. e^{-3}
6. $6.27e^{0.4}$
7. $11.9e^{-6}$
8. $e^{0.7419}$
9. $3.7e^{1.6263}$
10. $8.22e^{-1.3863}$

11. A capacitor discharges so that its terminal voltage is $v_C = 90e^{-200t}$. When $t = 0.004$, find (*a*) v_C and (*b*) dv_C/dt.

12. The primary current in a transformer is $i_1 = 6.4e^{-10t}$. The mutual inductance between the primary and secondary windings is $M = 2$ henrys. When $t = 0.001$, find (*a*) the primary current and (*b*) the induced secondary emf v_2.

13. The strength i of a telegraph signal decreases with distance s along a line in accordance with $i = 0.1e^{-0.15s}$. Find i when $s = 20$ miles.

14. In Prob. 11, after what time t will the voltage v_C be equal to 55 volts?

15. In the transformer of Prob. 12, to what value should the mutual inductance M be changed so that v_2 will be equal to 50 volts when $t = 0.001$ second?

16. At what distance along the telegraph line of Prob. 13 will the current be equal to 0.04 ampere?

17. A capacitor is connected through a resistance to a source of fixed potential V, so that the voltage across the capacitor at any instant can be shown to be $v_C = V(1 - e^{-t/RC})$. If $V = 100$ volts, and if $C = 0.004$ microfarad, find the value of R required for v_C to be 37.5 volts after 376 microseconds.

12-9 Integral of the reciprocal of a function. Suppose we are given that

$$y = \ln u \tag{20}$$

Then from (11), $dy/du = 1/u$, or

$$\frac{du}{u} = dy$$

Integrating, we have $\int du/u = y + C$, or, by (20)

$$\int \frac{du}{u} = \ln u + C \tag{21}$$

[When we want to integrate a *power function*, in general, we use the formula

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C$$

It will be recalled that this formula fails when $n = -1$; for there is no power of u which has the derivative u^{-1}. But we are now able to integrate even the reciprocal, or minus first power, of a function, by means of (21). Note that the result is not a power function, but a *logarithmic* function.]

Example 1. Evaluate $\int x \, dx/(x^2 + 1)$.

Note that if we let $u = x^2 + 1$, then $du = 2x \, dx$. Writing the given integral in the form

$$\int \frac{x \, dx}{x^2 + 1} = \frac{1}{2} \int \frac{2x \, dx}{x^2 + 1}$$

we find that it now has the form $\int du/u$. Then by (21),

$$\int \frac{x \, dx}{x^2 + 1} = \frac{1}{2} \ln (x^2 + 1) + C = \ln (x^2 + 1)^{1/2} + C$$

Example 2. Find $\int \tan x \, dx$.

This may be written

$$\int \frac{\sin x \, dx}{\cos x}$$

Multiplying and dividing by -1,

$$\int \tan x \, dx = - \int \frac{- \sin x \, dx}{\cos x}$$

But since $d(\cos x) = - \sin x \, dx$, this takes the form $\int du/u$, where $u = \cos x$. Then

$$\int \tan x \, dx = - \ln \cos x + C$$

or

$$\int \tan x \, dx = \ln \sec x + C \tag{22}$$

A formula which may be derived in a similar manner is

$$\int \cot x \, dx = \ln \sin x + C \tag{23}$$

the proof of which is left to you (Prob. 11).

Example 3. It was desired to find a relation between the received voltage V in a television receiver and the resulting "dot" brightness B on the screen, such that the interfering "noise" voltages would be least annoying.

It may be shown that the eye is least disturbed by extraneous impulses in the picture when the change in brightness ΔB resulting from a voltage change ΔV is

proportional to (*a*) the voltage change ΔV and (*b*) the existing brightness B. This desired condition is expressed

$$\Delta B = KB\,\Delta V$$

where K is a constant of proportionality. Consider the effect of allowing this voltage change ΔV to approach zero as a limit:

$$dB = KB\,dV \qquad \text{or} \qquad \frac{dB}{B} = K\,dV$$

Integrating, $$\ln B = KV + K_1$$

This is the desired relationship which should represent receiver operation. Experiment shows that, when the receiver functions according to such an equation, a pronounced improvement is effected in reducing the effects of noise in the picture. To correspond, the brightness-versus-output relation in a television station is specified by the Federal Communications Commission as the inverse of that in the equation just derived for the receiver.

PROBLEMS

In Probs. 1 to 10 integrate and check by differentiation.

1. $y = \int \frac{x\,dx}{3x^2 + 2}$

2. $y = \int \frac{x^2\,dx}{1 - 2x^3}$

3. $y = \int \frac{u^3\,du}{15 - u^4}$

4. $y = \int \frac{(1 - x)\,dx}{2x - x^2}$

5. $y = \int \frac{u^{1/2}\,du}{1 + u^{3/2}}$

6. $y = \int \frac{(x + \frac{1}{2})\,dx}{x^2 + x + 1}$

7. $y = \int \frac{e^u\,du}{1 - e^u}$

8. $y = \int \frac{\sec^2 u\,du}{1 + \tan u}$

9. $y = \int \frac{(x - 1/4\sqrt{x})\,dx}{x^2 - \sqrt{x}}$

10. $y = \int \frac{du}{u \ln u}$

11. Derive Formula (23): $\int \cot x\,dx = \ln \sin x + C$.

12. When light is absorbed in passing through a filter used with a television camera, its intensity I varies as a function of the thickness t of the filter as follows: $k\,dt = -(dI/I)$, where k is the absorption coefficient of the filter material (a constant). Integrating this expression, get a formula for the filter thickness which will reduce I to a given value.

13. In a certain loudspeaker horn, the cross-sectional area A varies as a function of length s according to $k\,ds = dA/A$. Find a formula for the length s at which A will have any given value.

14. The *magnetic potential* between two coaxial cylindrical shells is

$$M = ka \int_a^b \frac{dr}{r}$$

ampere-turns, where a and b are the radii of the cylinders and k is a constant. Carry out the integration, getting an equation for M.

15. The force in newtons on the piston of a certain engine varies with the distance, s meters, traveled by the piston according to $\mathbf{F} = 100/s$. Neglecting losses, how much energy is supplied by the piston in moving from $s = 0.1$ to $s = 0.2$?

16. A tapered open-wire transmission line is used to match two unequal impedances. The spacing A between the conductors varies with linear distance B along the line according to $K\,dB = dA/A$. What formula gives the distance B at which A will have any given value?

12-10 Quantities varying at constant relative rates.

Many times, a quantity varies at a *constant relative rate* (or *constant percentage rate*). That is, at any selected time the quantity is changing at a rate *proportional to its value* at that time. An example might be a voltage which at any instant decreases at a rate equal to 0.03 times its value at that instant:

$$\frac{dv}{dt} = -0.03v$$

In other words, the voltage is always changing at a rate which is a constant percentage of its value, in this case, the rate being always 3 per cent of the voltage.

More generally, suppose we are given that y is a function of time t, such that, after some instant designated as $t = 0$, the quantity y changes at a rate always equal to k times the value of y:

$$\frac{dy}{dt} = ky \tag{24}$$

Let us try to get a formula for y, and let it be desired that the formula contain no derivatives or differentials. We write (24) as

$$\frac{dy}{y} = k\,dt \qquad \text{or} \qquad \ln y = kt + C$$

It will prove convenient later to remember that the constant C must, of course, be the logarithm of some other constant, which we shall call K:

$$\ln y = kt + \ln K \tag{25}$$

$\ln y = kt \ln e + \ln K$

We now take antilogarithms of the quantities in (25). The antilogarithm of kt is e^{kt}. We remember that when we multiply two quantities a and b by means of logarithms, we *add* the logarithms. If

$$ab = c \qquad \text{then} \qquad \ln a + \ln b = \ln c$$

Thus in (25), the quantity having the logarithm $kt + \ln K$ must be the product of the quantity having the logarithm kt and the quantity having the logarithm $\ln K$. Thus

$$y = Ke^{kt} \tag{26}$$

It remains to evaluate the constant K. To accomplish this, we consider what conditions prevailed when $t = 0$. Making the substitution

$t = 0$ in (26), we find that when time was equal to zero, $y = K$. In other words, K in (26) is the value taken by y when $t = 0$ (called the *initial value* or *boundary value* of y). To indicate this fact, we shall write $y(0)$ instead of K. Then

$$\text{if} \quad \frac{dy}{dt} = ky \quad \text{then} \quad y = y(0)e^{kt} \tag{27}$$

where $y(0)$ is the initial value of y.

We note, then, the important fact that, if a function has a rate of change always proportional to its value, then it is an exponential function. It is so common to encounter functions of this kind that we prefer to remember the result (27) instead of going through the several steps of the foregoing derivations every time.

Example 1. The current i in a circuit changes at a rate which is $-2{,}000i$. If the current is initially 2.1 amperes, find the equation for i.

The given relations can be expressed

$$\frac{di}{dt} = -2{,}000i \qquad \text{and} \qquad i(0) = 2.1$$

Then by (27),

$$i = 2.1e^{-2{,}000t} \qquad \text{amperes}$$

Note that, when e is used as the base for expressing an exponentially changing quantity, the *constant relative rate* is clearly apparent in the exponent. The current found in Example 1, for instance, is always decreasing at a rate equal to $2{,}000i$, and the constant relative rate $-2{,}000$ appears explicitly in the exponent. But if the base had been made 10, for example, rather than e, the current relation would be, by (19),

$$i = 2.1(10)^{-(2{,}000 \ln 10)t} = 2.1(10)^{-4{,}605.2t}$$

In the latter form, the relative rate of change $-2{,}000$ is not apparent.

Sometimes we are not given the *initial* value of the function, but we may be given its value at some point other than that at which the independent variable is zero. In such cases, we *solve* for the initial value.

Example 2. The charge in a capacitor decreased at a rate always equal to 21 per cent of the amount of the charge. When t was equal to 1, the charge was 0.0002 coulomb. Find a formula for the charge.

The given rate can be written

$$\frac{dq}{dt} = -0.21q$$

By (27),

$$q = q(0)e^{-0.21t}$$

Substituting into this equation the given conditions $t = 1$, $q = 0.0002$, we get

$$0.0002 = q(0)e^{-0.21}$$
$$q(0) = 0.0002e^{0.21} = 0.000247 \text{ coulomb}$$

and
$$q = 0.000247e^{-0.21t} \quad \text{coulomb}$$

QUESTIONS

1. State a formula by which an exponential quantity using any base may be transformed to one using the base e.
2. State the formula for integrating the minus first power of a function (reciprocal of a function).
3. Given that $dy/dt = ky$ and that when $t = 0$ the corresponding value of y is some quantity $y(0)$, state an equation for y which is free of derivatives or differentials.

PROBLEMS

1. The intensity I of the light emitted from the phosphorescent spot of a cathode-ray tube decreased at a rate always proportional to I. State a formula for I as a function of t.

2. When the emf was removed from a series RL circuit, the voltage v_L across the inductor decreased at a rate equal to v_LR/L. What formula gives the value of v_L in terms of time?

3. The field intensity $\mathbf{E}$ of a radar beam decreases with distance s traversed by the beam. One cause of this decrease is absorption of the wave energy by the air. Neglecting other effects, this results in a rate of decrease always proportional to $\mathbf{E}$. Write an equation for $\mathbf{E}$ as a function of s.

4. Atomic fission results in a decrease in the amount M of a radioactive material. M changes at a rate equal to $-\delta M$, where t is the time in years and δ is the *decay constant* of the material. If $\delta = 4.1 \times 10^{-3}$, what formula expresses the amount of material remaining at any time from an original quantity of 1 milligram?

5. It is known that the voltage v_C across a certain capacitor, which is being discharged through a resistor, decreases at a rate equal to v_C/RC. State a formula for v_C at any time.

6. The intensity I of a sound in a broadcast studio died away at a rate equal to $27I$. State a formula expressing I as a function of time t in seconds.

7. The current i in a circuit had a rate of change always equal to $-12i$. When $t = 0$, i was 2 amperes. State an equation for i.

8. The difference T_D between the temperature of a heated conductor and that of the surrounding air changed after the heating current was shut off according to $dT_D/dt = -KT_D$. If T_D changes from 180 to 100 degrees in 50 seconds, find K.

9. The signal power P watts available from a wave traveling a distance s miles along a circuit changed as $dP/ds = -0.46P$. If $P = 0.004$ watt when $s = 1$ mile, find the value of P when $s = 10$ miles.

10. In undersea sound ranging absorption of sound energy in sea water is one of the causes of decrease in intensity I of a wave as it travels a distance s meters from the source. This absorption causes the rate of decrease in I to be equal to kI. For a certain wave $k = 7.84 \times 10^{-4}$. If I is 7.5×10^{-11} watt per square centimeter at a point 2,000 meters from the source, at what distance will it have a value of 7×10^{-12} watt per square centimeter? Neglect the effects of factors other than absorption.

12-11 Electric transients. In the previous parts of this chapter, you have been *given* certain exponential expressions as representing *transient* waveforms obtained when a voltage or current is suddenly applied to a circuit.* We now find that our calculus preparation allows us to *derive* and to study these expressions directly.

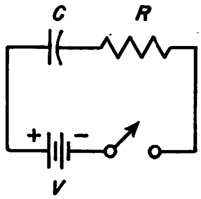


Fig. 12-5

First, consider the circuit of Fig. 12-5. Upon closing the switch at time $t = 0$, we get a charging current into the capacitor C through the resistor R. Assume that C is initially discharged. Kirchhoff's law states that the sum of the voltage drops across R and C must equal the battery voltage V:

$$Ri + \frac{q}{C} = V \tag{28}$$

Differentiating,

$$R\frac{di}{dt} + \frac{1}{C}i = 0 \qquad \text{since } \frac{dq}{dt} = i$$

or

$$\frac{di}{dt} = -\frac{1}{RC}i$$

This has the solution, by (27),

$$i = i(0)e^{-t/RC} \tag{29}$$

To know the initial current $i(0)$, we recall at once that, when $t = 0$, the voltage across C must be zero, since the voltage across a capacitor cannot change instantly from one value to another. Thus at $t = 0$ the entire battery voltage V must exist across R, making the initial current equal to V/R. The final current formula is then

$$i = \frac{V}{R}e^{-t/RC} \tag{30}$$

Thus, the current resulting when a battery charges a capacitor through a resistor is of the form of a decreasing exponential function. This form is graphed in Fig. 12-6. As a mathematical formality, it could be said that the capacitor theoretically never charges quite to the battery voltage, no matter how long we leave the circuit closed, for (30) will always indicate *some* current, no matter how large we make t. Actually, in a short time the charging current in most physical RC circuits reaches a value so small that we cannot measure it, since it becomes of the same order of magnitude as the random thermal movement of electrons in the circuit. We say that for practical purposes the capacitor is then charged to the source voltage.

* A *transient* may be considered a voltage or current waveform which exists during the time when a circuit is changing from one "steady" (ac or dc) condition to another.

It is of interest to have a simple way of expressing the "speed" of the transient wave of (30), that is, to be able to express the relative length of time needed for the current to fall to some certain proportion of its initial value. To approach this problem, let us first imagine that the current of Fig. 12-6 decreased continually at *its initial rate*, as shown by

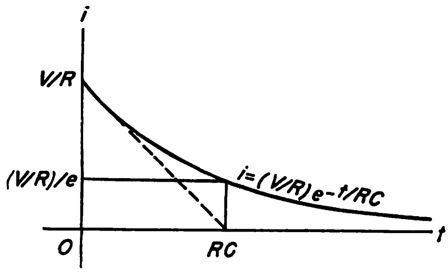


Fig. 12-6

the dotted line, instead of following the actual exponential curve. Then let us find how long it would take the current to decrease to zero at this rate. Differentiating (30),

$$\frac{di}{dt} = -\frac{V}{R^2C}e^{-t/RC} \tag{31}$$

Letting $t = 0$ in this expression, we find the initial rate of change of current to be $-V/R^2C$. At this rate, the time T required for the current to drop from V/R to zero would be

$$T = \frac{-V/R}{-V/R^2C} = RC \qquad \text{seconds} \tag{32}$$

The time T in (32) we call the *time constant* of a series RC circuit. It is numerically equal in seconds to the product RC (in ohms and farads, respectively).

Now let us find to what extent the *actual* current drops during the time interval T. Letting $t = T = RC$ in (30),

$$i = \frac{V/R}{e}$$

so that in time T seconds the actual current drops to a value equal to $1/e$ times its initial value. That is,

➤ The time constant of a series RC circuit is defined as the time required for a charging current from a dc source to decrease to a value equal to $1/e$ times its initial value. The time constant is numerically equal to $T = RC$ seconds.

The factor $1/e$ is approximately 0.368.

We now take up a different circuit. In Fig. 12-7, a battery having an internal resistance R_1 supplies a terminal voltage V to a series RL circuit, so that a steady current flows in the circuit.* Suppose the switch to be closed at time $t = 0$, removing the voltage V from the circuit. Let us find the manner in which the current decreases in the circuit RL.

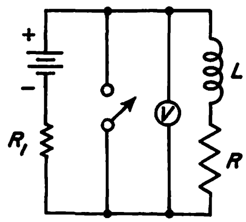


Fig. 12-7

By Kirchhoff's law, at any instant after $t = 0$

$$L\frac{di}{dt} + Ri = 0 \qquad \text{or} \qquad \frac{di}{dt} = -\frac{R}{L}i$$

By (27),

$$i = i(0)e^{-Rt/L} \tag{33}$$

To evaluate $i(0)$, we note that, prior to the closing of the switch, a steady current V/R flows in the circuit RL. This must also be the current when $t = 0$, since the current in an inductor cannot instantly change from zero to some other value. That is, $i(0) = V/R$, so that (33) becomes

$$i = \frac{V}{R}e^{-Rt/L} \tag{34}$$

The current, then, follows a decreasing exponential curve, just as in the charging of the RC circuit previously described. Here, too, the current theoretically never reaches zero; however, it actually drops to a negligible value in a comparatively short time in most circuits.

The *time constant* for the RL circuit may be defined similarly to that for the RC circuit:

The time constant of a series RL circuit is defined as the time required for an originally steady current to decrease to a value equal to $1/e$ times its original value after removal of the external emf. The time constant is numerically equal to $T = L/R$ seconds.

Calculation of the above values is left to you as a problem (Prob. 11).

QUESTIONS

1. What is a *steady state* in an electric circuit?
2. What is a transient waveform?
3. A certain capacitor was charged, through a resistor, by a dc source. In your own words, discuss the "length of time required to charge the capacitor to the source voltage."

* In duplicating this circuit, as for study with an oscilloscope, some external resistance should be added to R_1 to prevent the throwing of a short circuit across the battery when the switch is closed.

4. Define the *time constant* of an RC circuit.
5. Define the *time constant* of an RL circuit.
6. Discuss the meaning of a statement by an electronics worker that the current in a certain RL circuit "decreased to zero in 0.1 second after the voltage was removed."

PROBLEMS

1. Find the time constants of the following circuits: (*a*) the RC circuit for the avc system of a broadcast receiver using a 2-megohm resistor and a capacitor of 0.1 microfarad; (*b*) the differentiating circuit of a television receiver having a capacitor of 1,500 micromicrofarads and a resistor of 680 ohms; (*c*) the agc circuit of a television receiver having a capacitor of 0.5 microfarad and a resistor of 27,000 ohms; (*d*) the "fast-time-constant" circuit of a radar, where a 51-micromicrofarad capacitor is used with a 20,000-ohm resistor; (*e*) a television integrating circuit, where a resistor of 100,000 ohms is used with a capacitor of 0.001 microfarad; (*f*) a series circuit consisting of a 120-millihenry inductor and a 330-ohm resistor.

2. (*a*) A series RC circuit is to have a time constant of 155 microseconds. If $C = 0.01$ microfarad, find the required value of R. (*b*) In order to have a time constant of 100 microseconds, what value of capacitance should an RC circuit contain, where $R = 33{,}000$ ohms?

3. (*a*) Applying Ohm's law, find an expression for the voltage v_R across the resistor of Fig. 12-5 as the charging of C progresses. (*b*) Applying Kirchhoff's voltage law, use the result of (*a*) to show how the voltage v_C across C of Fig. 12-5 increases. (*c*) From the result of (*b*), what will be the value of v_C in terms of V after a time interval equal to the time constant T of the circuit? (*d*) Write an expression for the charge q in coulombs which will be stored in the capacitor at any time t.

4. In the circuit of Fig. 12-5 let $C = 0.02$ microfarad, $R = 20{,}000$ ohms, and $V = 100$ volts. (*a*) What is the time constant of the circuit? Make the following calculations, assuming that the switch has been closed for 100 microseconds: (*b*) Find the current i. (*c*) Find the voltage across R. (*d*) Find the voltage across C. (*e*) Find the charge q contained in C.

5. In the circuit of Prob. 4 plot several values of the voltage v_C across the capacitor corresponding to various times t. Sketch a graph showing how this voltage rises with time.

6. Applying the results of Prob. 3, find (*a*) what the voltage across a capacitor of 0.05 microfarad will be when it has been charged from a 200-volt source through a 1-megohm resistor for 40 milliseconds; (*b*) after what time the voltage across the capacitor in part (*a*) will be equal to 70 volts.

7. A capacitor is charged to a voltage V. It is then discharged through a resistance R. Derive an equation for the discharge current as a function of time. Compare with (30).

8. (*a*) By Ohm's law, find from the result of Prob. 7 an equation for the voltage v_R across the resistor R in that problem at any time t. (*b*) From (*a*) determine an equation for the capacitor voltage v_C.

9. An RC circuit has a time constant of 50 microseconds. If the capacitor is first charged to a voltage of 100 volts and the RC circuit is then closed upon itself, after what time will the capacitor voltage be equal to 10 volts?

10. (*a*) Using Ohm's law, get an equation for the voltage v_R across the resistor of the RL circuit of Fig. 12-7 at any time t after the switch is closed. (*b*) From (*a*) determine an equation for the inductor voltage v_L.

11. In Fig. 12-7 find (*a*) the rate of change of i with respect to t after the closing of

the switch; (*b*) the time T which would be required for i to decrease to zero if it continued to decrease at the initial rate. Defining T as the *time constant* of this circuit, find (*c*) the value of i in terms of V/R after an interval of one time constant.

12. In Fig. 12-7, if $L = 0.88$ henry, $R = 220$ ohms, and $V = 1.8$ volts, find (*a*) an equation for the current i in the LR circuit; (*b*) the time constant T of the LR circuit (by Prob. 11); (*c*) the current after 3 milliseconds; (*d*) the time at which i will be equal to 5 milliamperes.

12-12 The Weber-Fechner law.

Studies have been made to find the effect upon the senses of a change in the strength of an external stimulus. As applied to the sense of *hearing*, some of the results can be stated as follows:

➡ The change ΔI in sound intensity required to produce a given change in loudness ΔL is approximately proportional to (*a*) the desired loudness change ΔL and (*b*) the sound intensity I already existing (Weber's law).*

(Note that the *intensity* of a sound indicates its strength, as measured by physical means, while *loudness* is an indication of the effect of the sound upon the senses.)

Stated as an equation, the above law becomes

$$\Delta I = KI\,\Delta L \tag{35}$$

Some mathematical operations upon the Weber law give these results: first, let it be desired to make a *barely perceptible* change in intensity, that is, to let ΔL approach zero as a limit. We get

$$dI = KI\,dL \qquad \text{so that} \qquad \ln I = KL + K_1$$

This may be written

➡
$$L = C \ln I + K_2 \tag{36}$$

where $C = 1/K$ and $K_2 = -K_1/K$. We proceed to find the significance of K_2 and of C. As we shall show, K_2 is influenced by the choice of units in which I is measured. Consider a sound whose intensity is just small enough so that the sound would become inaudible with any further decrease. We let the symbol $I(0)$ represent this value of intensity, and we say that the loudness of the sound is zero. Then, for this sound, (36) becomes

$$K_2 = -C \ln I(0)$$

* This law and the results shown in the rest of this section are only approximate; the actual effects vary from one individual to another and even from time to time in a single individual. However, the law is of sufficient accuracy to give useful results. It is due to E. H. Weber of Germany (1795–1878). For data obtained in more recent and detailed studies, the interested reader should see ref. 1.

Substituting this value in (36),

$$L = C \ln \frac{I}{I(0)} \tag{37}$$

The intensity $I(0)$ at which a 1,000-cycle-per-second sound becomes inaudible has been determined approximately, by measurements upon many individuals, as 10^{-16} watt per square centimeter.

If we should hypothetically measure I in terms of a unit whose size was exactly $I(0)$, then the denominator in (37) would become unity. This would make (37) read, in such a special case,

$$L = C \ln I \tag{38}$$

Thus, the statement is often made that

➢ The loudness of a sound is approximately proportional to the logarithm of its intensity (Fechner's law).*

The quantity C in (38) is a constant of proportionality. Its value may be estimated from experimental results, and it indicates the *slope* of a graph relating L to $\ln I$.

Equation (37) may be written

$$L = \frac{C}{M} \log_{10} \frac{I}{I(0)} \tag{39}$$

But the *intensity level* of a sound is defined as†

$$\text{Intensity level} = 10 \log_{10} \frac{I}{I(0)} \qquad \text{decibels} \tag{40}$$

From (39) and (40), the loudness of a sound is

➢ $$L = \frac{C}{10M} \times (\text{intensity level in decibels}) \tag{41}$$

That is,

➢ The loudness of a sound is approximately proportional to its intensity level in decibels

(limited by inaccuracies in the Weber-Fechner law). This fact contributes to the practical usefulness of the decibel as a unit.

Similar results are obtained when the sense of *vision* is considered. It can be shown that, over a large part of the visible brightness range,

* Due to the German scientist G. T. Fechner (1801–1887). Both Weber's law and Fechner's law are often referred to as the Weber-Fechner law; they are essentially equivalent.

† Here, as is often done, we have taken the reference intensity as $I(0)$, the accepted minimum audible intensity at 1,000 cycles per second.

the response R of the eye is approximately related to the brightness B of a source by

$$R = C \ln B \tag{42}$$

where C is a constant [compare with (38), for the sense of hearing].

Example. Find a relation between the brightness B_2 of the *element* (dot) in the image on a television screen and the brightness B_1 of the corresponding element in the actual scene being televised, such that an illusion of realism will be obtained.

One possible relation would be such that the brightness of each image element would be *equal* to the brightness of the corresponding element in the televised object. But other relations will also serve, as will be shown. A realistic impression will be had by the viewer if the response R_2 of the eye to an element on the receiver screen is made proportional to the visual response R_1 of a viewer in the studio to the corresponding element in the actual scene. In symbols, the relation

$$R_2 = \gamma R_1 \tag{43}$$

will provide this relation. Here, γ is some constant of proportionality. Applying (42), we can write (43) thus:

$$\ln B_2 = \gamma \ln B_1 \qquad \text{or} \qquad B_2 = B_1{}^{\gamma} \tag{44}$$

A further modification can now be made to provide for the fact that the maximum brightness of the received image is generally different by some factor k from the maximum brightness of the actual scene. Applying this modification to (44), we have

$$B_2 = kB_1{}^{\gamma} \tag{45}$$

as a general relation which provides realistic results. This is sometimes written

$$\log_{10} B_2 = \gamma \log_{10} B_1 + \log_{10} k \tag{46}$$

(The quantity γ is a controlling factor in the appearance of *contrast* in the picture. In black-and-white television γ is generally made to have a value somewhat greater than unity to compensate for lack of *color contrast* in the image.)

QUESTIONS

1. State approximately the way in which the loudness of a tone varies as a function of its intensity.
2. What approximate relation exists between the loudness of a sound and its intensity level in decibels?
3. In what approximate way is the response of the visual sense related to the brightness of an object?
4. The statement is sometimes made that "the ear is sensitive to *percentage* changes in sound intensity." Discuss this statement in the light of (35). What is an analogous statement regarding the visual sense?

PROBLEMS

1. To produce an appreciable improvement in the loudness of a program from a public-address amplifier, it was necessary to increase the power output from 10 to 20 watts. To what value should the output power be increased to obtain a second similar improvement?

2. It was found that the power output P of a radio-telegraph transmitter had to be increased from 10,000 to 14,900 watts to produce a given change in loudness of the signal for a receiving operator. If it could be assumed that the intensity of the received sound was proportional to P, to what value would P have to be increased to provide a second similar increase in loudness, according to (35)?

3. Listeners reported a certain decrease in the loudness of a program when the volume control setting was reduced by 4.5 decibels. According to (41), what change in setting would result in a second similar decrease in loudness?

4. If a series of experiments gives the result in (36) that $C = 2{,}230$ and $K_2 = 13{,}300$ over a given range of intensities for a certain tone, find dL/dI in loudness units per watt per square centimeter when $I = 6 \times 10^{-11}$ watt per square centimeter.

5. According to one set of experimental results, the value of C in (42) is 30 over a given range of brightness values. Find a formula for dR/dB.

12-13 Integral of the exponential function.

Equation (17) gives

$$\frac{d}{dx}\,a^u = a^u \frac{du}{dx} \ln a$$

or

$$a^u\,du = d(a^u)\,\frac{1}{\ln a}$$

which, when integrated, gives

$$\boxed{\int a^u\,du = \frac{a^u}{\ln a} + C} \qquad (47)$$

If the given exponential is expressed to the base e, we have

$$\boxed{\int e^u\,du = e^u + C} \qquad (48)$$

(Note the interesting result that

An exponential e^u is not only its own derivative with respect to u but, within a constant, is also its own integral with respect to u.)

Example 1. Integrate $\int e^{2x}\,dx$.

Multiplying and dividing by 2, we convert the given integral to $\frac{1}{2}\int e^{2x}\,2dx$, in which $2dx$ is the differential of the exponent $2x$. This is of the form $\int e^u\,du$, so that

$$\int e^{2x}\,dx = \tfrac{1}{2}\int e^{2x}\,2dx = \tfrac{1}{2}e^{2x} + C$$

Example 2. The current in a circuit varied according to $i = (5 \times 10^{-3})te^{t^2}$. Find the charge transferred by the current in 1 second.

Since $q = \int i\,dt$, we are to evaluate

$$q = (5 \times 10^{-3}) \int_0^1 te^{t^2}\,dt$$

If we designate the exponent t^2 as u, then $du = 2t\,dt$. We therefore multiply and divide the given integral by 2 and write

$$q = \tfrac{1}{2}(5 \times 10^{-3}) \int_0^1 e^{t^2} 2t\,dt$$

The expression under the integral sign is now of the form $e^u\,du$. Applying (48),

$$q = (2.5 \times 10^{-3})e^{t^2}\Big]_{t=0}^{1} = (2.5 \times 10^{-3})(e - 1) = 4.3 \times 10^{-3} \text{ coulomb}$$

QUESTION

1. State formulas for the following integrals: (a) $\int a^u\,du$ (b) $\int e^u\,du$

PROBLEMS

In Probs. 1 to 10 integrate and check the results by differentiation.

1. $\int e^{-3x}\,dx$
2. $\int 3^x\,dx$
3. $\int 10^{-5x}\,dx$
4. $\int 4^{2t}\,dt$
5. $\int e^{-x/2}\,dx$
6. $\int \exp(2x + 2)\,dx$
7. $\int e^{\sin x} \cos x\,dx$
8. $\int x^2 \exp(x^3 + 7)\,dx$
9. $\int (x + 1) \exp(x^2 + 2x)\,dx$
10. $\int (10x^2 + 5) \exp(2x^3 + 3x)\,dx$

11. The current supplied to a capacitor was $i = (V/R)e^{-t/RC}$, where t was in seconds. If $V = 200$ volts, $R = 10{,}000$ ohms, and $C = 0.002$ microfarad, find the charge delivered to the capacitor from $t = 0$ to $t = 5$ microseconds.

12. The voltage across an inductor varied according to $v = 200e^{-50t}$. Find the average voltage across the inductor, taken over the interval from $t = 0$ to $t = 0.03$ second.

13. The voltage across a 0.01-microfarad capacitor changed at a rate $dv_C/dt = -0.001e^{-0.0001t}$. If $v_C = 10$ when $t = 0$, find a formula for the charge in coulombs at any time t.

14. The signal voltage v across a telephone circuit changed with distance s miles along the circuit at a rate $dv/ds = -0.36e^{-0.12s}$. If the applied voltage at the sending end was 3 volts, at what distance along the line was the voltage equal to one-half that value?

15. A current $i = 0.4e^{-500t}$ flowed in a 2,000-ohm resistor. Find the energy loss in the resistor during the interval from $t = 0$ to $t = 0.003$ second.

16. During a certain interval, a charged particle had an acceleration $\mathbf{a} = 2p^2e^{pt}$, where p was a constant. Find formulas for (a) its speed and (b) the distance which it traveled.

17. The cross-sectional area (A square feet) of a speaker horn varies with distance (s feet) along its axis according to $A = 1.1e^{0.7s}$. Find the volume of the horn if its length is 5 feet.

18. An "exponential" transmission-line section had a spacing (y meters) between conductors which changed as a function of distance (x meters) along the line according to $y = 0.092e^{0.52x}$. If the section was 1 meter long, find the area between the conductors.

19. After a certain time $t = 0$, an electron moved over a portion of its path so that its speed was $ds/dt = 1{,}000e^{7t}$ meters per second. Find the distance which it traveled from $t = 0$ to the time when its speed was 5,000 meters per second.

20. The current in a 1,000-ohm resistor varied according to $i = 0.5(1 - e^{-1{,}000t})$. What was the average power in the resistor over the interval from $t = 0$ to $t = 0.002$ second?

REFERENCE

1. L. L. Beranek: "Acoustic Measurements," John Wiley & Sons, Inc., New York, 1949.

13

Hyperbolic Functions

We next take up *hyperbolic functions,* a class of functions which are of great usefulness in electrical problems. These functions are defined in terms of the exponential functions treated in the preceding chapter, but they have many features in common with the trigonometric functions of Chap. 11.

13-1 The hyperbolic cosine. Consider the function

$$y = \frac{e^u + e^{-u}}{2} \tag{1}$$

Functions of this kind occur often in electrical studies. Note that this function involves the sum of two *exponential* functions. It is graphed in Fig. 13-1.

To illustrate an application of (1), consider an antenna wire suspended as shown in Fig. 13-2. If the wire is flexible and nonstretching, it may be shown that its height h feet varies with horizontal distance x feet from the center point according to some such equation as

$$h = 100\frac{e^{0.01x} + e^{-0.01x}}{2} - 50 \tag{2}$$

Figure 13-2 has been drawn to illustrate a wire whose height varies according to (2).

Observe that the fractional quantity in (2) has the form of the function presented in (1), where $u = 0.01x$. For reasons to be discussed shortly,

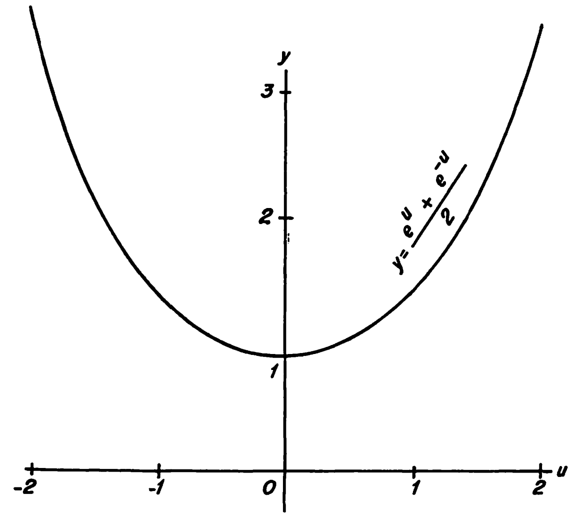


Fig. 13-1

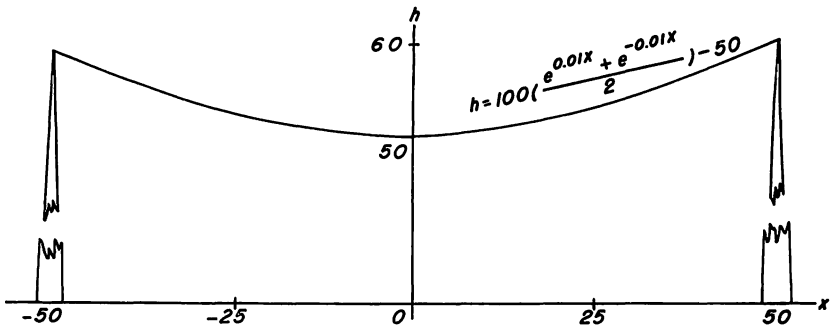


Fig. 13-2

we refer to the function (1) as the *hyperbolic cosine of u* (symbolized by cosh u).* That is,

$$\cosh u = \frac{e^u + e^{-u}}{2} \tag{3}$$

It would, then, be proper to write (2) in the form

$$h = 100 \cosh 0.01x - 50 \tag{4}$$

* For brevity, this is often read aloud "cose h of u" or, in some classes, "cosh u."

Since the function (3) is called a hyperbolic *cosine* it is well to compare, and to contrast, certain of its properties with those of the trigonometric (or circular) cosine function studied in Chap. 11. The following points should be noted:

1. It may be shown that the function cosh u is related to the curve known as an *hyperbola* in much the same way that cos u is related to a *circle;* and, for this reason, the name *hyperbolic* has been attached to the function. This property is, however, neither the most interesting nor the most useful characteristic of the function cosh u, so we do not give it further consideration here. References are provided for those who wish to consider this point further.[1,2]

2. Although the function (3) is called a hyperbolic *cosine*, it is *not* a periodic function; that is, it does not repeat its values cyclically as u increases, as the circular function cos u does.

13-2 Other hyperbolic functions. Further functions of the hyperbolic variety are defined in terms of exponentials, as follows.

The *hyperbolic sine of u* (sinh u) is defined as

$$\sinh u = \frac{e^u - e^{-u}}{2} \tag{5}$$

The ratio of sinh u to cosh u is called the *hyperbolic* tangent of u (tanh u):

$$\tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}} \tag{6}$$

The reciprocal of the relation (6) is called the *hyperbolic cotangent of u* (coth u or, sometimes, ctnh u):

$$\coth u = \frac{e^u + e^{-u}}{e^u - e^{-u}} \tag{7}$$

The *hyperbolic secant of u* (sech u) and the *hyperbolic cosecant of u* (csch u) are the reciprocals of cosh u and of sinh u, respectively:*

$$\operatorname{sech} u = \frac{1}{\cosh u} = \frac{2}{e^u + e^{-u}} \tag{8}$$

$$\operatorname{csch} u = \frac{1}{\sinh u} = \frac{2}{e^u - e^{-u}} \tag{9}$$

It is seen that the *hyperbolic* tangent, cotangent, secant, and cosecant are related to the hyperbolic sine and cosine in the same ways that the

* In discussing the foregoing functions aloud, it is convenient to say "sine *h* of u" for sinh u; "tan *h* u" for tanh u; "co-tan *h* u" for coth u; "seek *h* u" for sech u, and "co-seek *h* u" for csch u. Some classes say "shin u" for sinh u; "than u" for tanh u; "coth u" for coth u; "sech u" for sech u; and "co-sech u" for csch u. In place of sinh u, cosh u, etc., we sometimes see printed in German letters the equivalents of the corresponding *circular*-function abbreviations: 𝔖𝔦𝔫 u, ℭ𝔬𝔰 u, etc.

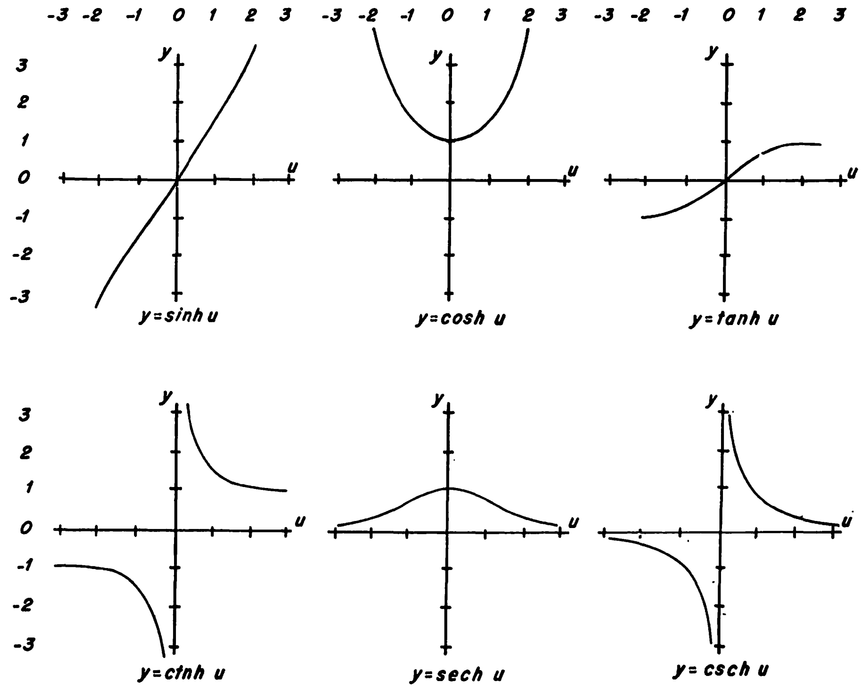


Fig. 13-3

corresponding *circular* functions are related to the trigonometric sine and cosine functions.

The six hyperbolic functions which we have defined are graphed in Fig. 13-3. The two facts concerning the hyperbolic cosine which were listed at the end of Sec. 13-1 could be restated to apply as well to the remaining five hyperbolic functions just described.

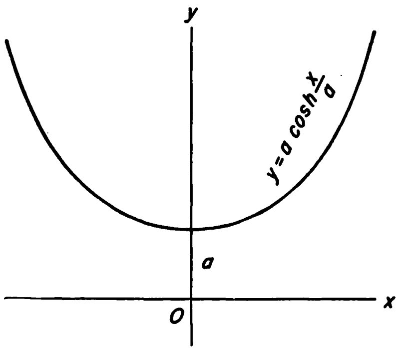


Fig. 13-4

13-3 The catenary. It can be shown* that a flexible cable having uniform weight per unit length, when suspended from two points of equal height, will hang in a curve given by

$$y = a \cosh \frac{x}{a} \qquad \textbf{(10)}$$

(Fig. 13-4). Here a is a constant, and x and y are, respectively, the horizontal and vertical coordinates of any point P on the cable referred

* See ref. 1, pp. 195–199.

to an origin O located at a distance a below the lowest point on the cable. The curve defined by (10) and pictured in Fig. 13-4 is called a *catenary*. Among its uses is the design of power and communication cable systems.

It turns out that the tension at any point on the cable is equal to $T = wy$, where w is the weight of the cable per unit length. The horizontal component of the tension at any point on the cable is $H = aw$.

QUESTIONS

1. State in terms of exponentials the definitions of the following:
(*a*) sinh u (*b*) cosh u (*c*) tanh u (*d*) coth u (*e*) sech u (*f*) csch u

2. State the relations of the following functions to either sinh u or cosh u or both:
(*a*) tanh u (*b*) coth u (*c*) sech u (*d*) csch u

3. Of the six hyperbolic functions which we have defined, which, if any, are periodic?

PROBLEMS

1. Look up in the tables of exponential functions the values of $e^{0.5}$ and of $e^{-0.5}$. By (3), calculate the value of cosh 0.5. Compare with the value in Appendix Table 6.

2. Using the information already looked up for Prob. 1, calculate by (5) the value of sinh 0.5. Compare with Table 6.

3. Find the height of the antenna whose height is given by (4) at a distance x feet from the center (*a*) when $x = 20$ and (*b*) when $x = 50$.

4. Each of the series arms of a symmetrical T attenuator which connects circuits of impedance $\boldsymbol{Z}$ should have a resistance $R_1 = \boldsymbol{Z} \tanh (N/2)$, where N is the desired attenuation in nepers. Find R_1 if $\boldsymbol{Z} = 600$ ohms and $N = 1.2$ nepers.

5. The input current to a certain telegraph wire of length s miles is $I = (V/R_0) \coth 0.004s$, where V is the applied voltage and R_0 is the characteristic impedance of the circuit. Find I if $V = 180$ volts, $R_0 = 310$ ohms, and $s = 35$ miles.

6. Under appropriate conditions, the voltage v across a transmission line at a distance s meters from the transmitting end is

$$v = 110 \cosh (5 \times 10^{-6}\, s) - 99 \sinh (5 \times 10^{-6}\, s)$$

Find v when $s = 90{,}000$.

7. A transmission line consists of a conductor of diameter d centered at a distance h from the bottom of a rectangular metallic trough of width w. Under certain conditions the characteristic impedance of the line is $\boldsymbol{Z}_0 = 60 \ln [4w \tanh (\pi h/w)/\pi d]$. Find $\boldsymbol{Z}_0$ if $d = 1$ centimeter, $h = 3$ centimeters, and $w = 4$ centimeters.

8. An antenna wire hangs according to the catenary curve $y = 50 \cosh 0.02x$ feet. Its supporting poles are 80 feet apart. If the wire weighs 3.14 pounds per hundred feet, find (*a*) the tension in the wire at a point 20 feet from one of the poles and (*b*) the upsetting (horizontal) component of the force applied to each of the poles.

13-4 Derivatives of the hyperbolic functions. To obtain a differentiation formula for sinh u, we first write this function in its *exponential* form

$$\sinh u = \frac{e^u - e^{-u}}{2} = \frac{e^u}{2} - \frac{e^{-u}}{2}$$

Differentiating term by term,

$$\frac{d}{dx}\sinh u = \frac{e^u}{2}\frac{du}{dx} + \frac{e^{-u}}{2}\frac{du}{dx} = \frac{e^u + e^{-u}}{2}\frac{du}{dx}$$

or

$$\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx} \tag{11}$$

If we differentiate cosh u, starting with its exponential form, we get

$$\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx} \tag{12}$$

To obtain the derivative of tanh u, we write this function

$$\tanh u = \frac{\sinh u}{\cosh u}$$

which, differentiated as a quotient, yields

$$\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx} \tag{13}$$

The derivative of coth u is similarly obtained:

$$\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \frac{du}{dx} \tag{14}$$

Next, we differentiate sech u:

$$\frac{d}{dx}\operatorname{sech} u = \frac{d}{dx}(\cosh u)^{-1} = -\cosh^{-2} u \sinh u \frac{du}{dx}$$

which may be arranged

$$\frac{d}{dx}\operatorname{sech} u = -\frac{1}{\cosh u}\frac{\sinh u}{\cosh u}\frac{du}{dx}$$

or

$$\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx} \tag{15}$$

A similar treatment of csch u gives

$$\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx} \tag{16}$$

It is convenient to note that the six differentiation formulas just obtained are parallel to those for the trigonometric functions, with the exception of the *signs* of some of the derivatives.

QUESTIONS

1. State the equation defining the curve called a *catenary*.
2. Under what conditions will a cable hang in the form of a catenary?
3. Give the formula for differentiating (*a*) sinh u; (*b*) cosh u; (*c*) tanh u; (*d*) coth u; (*e*) sech u; (*f*) csch u.

PROBLEMS

In Probs. 1 to 10 differentiate with respect to x.

1. $y = \sinh (x/5)$
2. $y = \cosh (5x - 3)$
3. $y = \cosh x^2$
4. $y = \tanh (1 - x^2)$
5. $y = \coth x^{1/2}$
6. $y = \operatorname{sech} (x^2 - x)$
7. $y = \operatorname{csch} (x + \frac{1}{2}x^2)$
8. $y = \sinh^{3/2} x^2$
9. $y = \sin x \sinh x$
10. $y = e^{\cosh x} \cosh x$

11. The series arm of a π-section resistive attenuator has a value $R_3 = \boldsymbol{Z} \sinh N$, where $\boldsymbol{Z}$ is the circuit impedance and N the desired loss in nepers. Find dR_3/dN.

12. The current in a certain transmission line varied with distance s meters from the sending end according to $i = 0.060 \cosh (4 \times 10^{-6}s) - 0.054 \sinh (4 \times 10^{-6}s)$. Find di/ds when $s = 150{,}000$ meters.

13. The height h feet of a cable above earth at a horizontal distance x feet from the center point is $h = 200 \cosh 0.005x - 155$. Find the slope of the cable at a distance 100 feet from the center point.

14. The input series arm of an unbalanced T attenuator has a resistance $R_1 = \boldsymbol{Z}_1 \coth N - R_3$, where $\boldsymbol{Z}_1$ is the source impedance, N is the loss in nepers, and R_3 is the corresponding shunt arm of the attenuator. For a given $\boldsymbol{Z}_1$ and R_3, how fast does R_1 change with respect to N?

15. The primary current in a transformer changed according to $i_1 = 0.012 \sinh 1.15t$ amperes. If the mutual inductance between the primary and secondary windings was 3.25 henrys, find the emf v_2 induced in the secondary when $t = 1.9$ seconds.

16. A voltage V is applied to a telegraph line of length s miles, resulting in a current at the output point equal to $I = V/(R_0 \sinh 0.045s)$. Find dI/ds.

17. The charge taken from a capacitor was $q = 0.2 \cosh 0.06t$ coulombs. What was the discharge current when $t = 2$ seconds?

18. The shunt arm of a symmetrical resistive T pad has a resistance $R_3 = \boldsymbol{Z} \operatorname{csch} N$, where $\boldsymbol{Z}$ is the circuit impedance and N the loss in nepers. If $\boldsymbol{Z} = 500$ ohms, find dR_3/dN when $N = 2.1$.

19. A parallel-wire transmission line has conductors of diameter c spaced a distance b between centers. The line is located midway between two metal plates a distance h apart. Under certain conditions the impedance of the cable is

$$\boldsymbol{Z}_0 = 120 \ln \left[\frac{4h \tanh (\pi b/2h)}{\pi c} \right]$$

At what rate does $\boldsymbol{Z}_0$ change with respect to h?

13-5 Some relations involving hyperbolic functions.

Often we must find some particular relation involving hyperbolic functions. Sometimes the desired relation is an identity which relates certain hyperbolic functions to each other, and sometimes it is necessary to find a connec-

tion between hyperbolic functions and other kinds of functions. In performing these operations we shall often find it convenient to express *hyperbolic* functions in their exponential forms. At other times we may profit by putting *exponential* functions into hyperbolic forms. We should refer, where necessary, to the table of Relations Involving Hyperbolic Functions (Table 7 of the Appendix). Some examples of varying degrees of difficulty follow.

Example 1. Find the value of $\cosh^2 u - \sinh^2 u$.

This result is given in Table 7, but we derive it here for practice purposes. In exponential forms,

$$\cosh^2 u - \sinh^2 u = \left(\frac{e^u + e^{-u}}{2}\right)^2 - \left(\frac{e^u - e^{-u}}{2}\right)^2$$
$$= \frac{e^{2u} + 2 + e^{-2u}}{4} - \frac{e^{2u} - 2 + e^{-2u}}{4}$$

Clearing of fractions gives

$$\cosh^2 u - \sinh^2 u = 1 \tag{17}$$

(Observe that this identity is similar, except in signs, to the trigonometric identity $\sin^2 u + \cos^2 u = 1$. A large number of trigonometric identities have their parallels in hyperbolic identities.)

Example 2. In studying an electric transient it was necessary to simplify $\sqrt{A} \cosh x - B \sinh x = 0$.

If we divide each term by the negative of the first,

$$-1 + \frac{B \sinh x}{\sqrt{A} \cosh x} = 0 \qquad \text{or} \qquad \tanh x = \frac{\sqrt{A}}{B}$$

Example 3. Show that

$$e^x = \cosh x + \sinh x \tag{18}$$

This formula, also given in Table 7, is readily shown by writing the hyperbolic functions in their exponential forms:

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

Placing the right member over a common denominator 2, we get $e^x = e^x$. Since this reduces to the identity

$$1 = 1$$

the given formula is proved. A similar method can be used to prove that

$$e^{-x} = \cosh x - \sinh x \tag{19}$$

Example 4. In solving for the current in a circuit, it was found that $i = Ae^t + Be^{-t}$, where A and B were constants. Express this result in terms of hyperbolic functions.

The given current form can be expressed variously in terms of hyperbolic functions. By (18) and (19),

$$Ae^t = A \cosh t + A \sinh t$$

and $$Be^{-t} = B \cosh t - B \sinh t$$

so that $$i = (A + B) \cosh t + (A - B) \sinh t$$

Let $M = A + B$, $N = A - B$:

$$i = M \cosh t + N \sinh t \tag{20}$$

In case $M^2 > N^2$, multiply and divide the right member by $(M^2 - N^2)^{1/2}$:

$$i = \sqrt{M^2 - N^2}\left(\frac{M}{\sqrt{M^2 - N^2}} \cosh t + \frac{N}{\sqrt{M^2 - N^2}} \sinh t\right)$$

Introduce a quantity ϕ, such that $M = \cosh \phi$ and $N = \sinh \phi$. By (17), $(M^2 - N^2)^{1/2} = 1$:

$$i = \cosh t \cosh \phi + \sinh t \sinh \phi \tag{21}$$

This is one form of hyperbolic expression for i. If we add relations 19 and 20 of Table 7 we get

$$\cosh x \cosh y + \sinh x \sinh y = \cosh (x + y)$$

This makes (21) above read

$$i = \cosh (t + \phi)$$

PROBLEMS

1. Prove the identity $\tanh^2 u + \operatorname{sech}^2 u = 1$.

2. Prove the identity $\coth^2 u - \operatorname{csch}^2 u = 1$.

3. The voltage in a circuit was $v = 10(e^t - e^{2t})$. Show that this is equivalent to a voltage $v = -20e^{3t/2} \sinh (t/2)$. (HINT: Set these two forms equal and show that the resulting equation reduces to an identity, $1 = 1$).

4. Prove Formula (19).

5. The current in a circuit varied according to $i = 0.5e^{0.002t}$. Show that this current may be considered as the sum of two other currents, as expressed by $i = 0.5 \sinh 0.002t + 0.5 \cosh 0.002t$. [Use (18).]

6. Prove the identity $\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y$. (HINT: Show that the given equation reduces to the form $1 = 1$.)

7. In case $M^2 < N^2$ in (20), multiply and divide the right member by $(N^2 - M^2)^{1/2}$. Then introduce a new variable θ, such that $M = \sinh \theta$ and $N = \cosh \theta$. Show that in this case (a) $i = \cosh t \sinh \theta + \sinh t \cosh \theta$ and (b) through a suitable identity from Table 7 that an equivalent form is $i = \sinh (t + \theta)$.

8. The current in a circuit varied as $i = e^{0.2t} + 2e^{-0.2t}$. Show that this is equivalent to $i = 3 \cosh 0.2t - \sinh 0.2t$.

13-6 Inverse hyperbolic functions.

Suppose that

$$u = \sinh y$$

Other ways of stating this relation are

$$y = \sinh^{-1} u \quad \text{or} \quad y = \operatorname{arcsinh} u \quad \text{or} \quad y = \operatorname{arg\,sinh} u \tag{22}$$

Equation (22) is read aloud: "y is equal to the inverse hyperbolic sine of u" or sometimes "y is the quantity whose hyperbolic sine is u."*

Other inverse hyperbolic functions are written in a corresponding manner; $\coth^{-1} u$ (or $\text{ctnh}^{-1} u$), for instance, indicating the inverse hyperbolic cotangent of u (the quantity whose hyperbolic cotangent is u).

Figure 13-5 shows the graphs of various inverse hyperbolic functions.

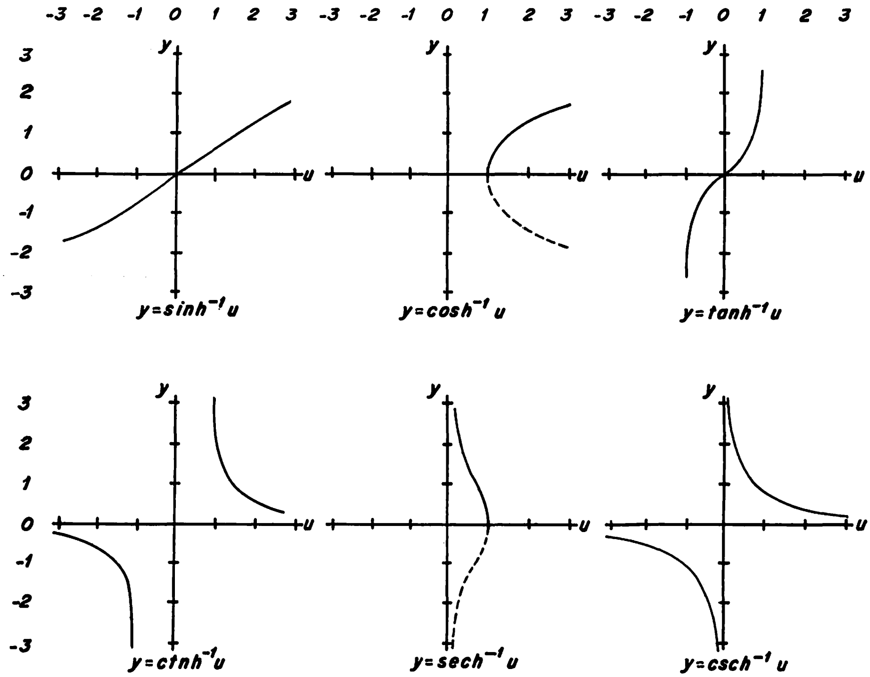


Fig. 13-5

Comparison with Fig. 13-3 brings out at once the mirror-reflection relationship between the graphs of the direct and the inverse hyperbolic functions with respect to a line $y = u$, just as in the case of the trigonometric and the logarithmic functions and their inverse functions.

As seen in Fig. 13-5, the function $\cosh^{-1} u$ has two possible values for each value of u having a hyperbolic cosine (except at $u = 1$). A similar situation applies to $\text{sech}^{-1} u$. We arbitrarily take the upper or *positive* ranges of these functions as their ranges of *principal values.*

* The form $\sinh^{-1} u$ is recommended for the inverse hyperbolic sine of u. For brevity, we may read (22) aloud "y is equal to the inverse sine h of u," or "y is equal to the inverse shin of u." Similarly for other inverse hyperbolic functions.

13-7 Derivatives of the inverse hyperbolic functions. To differentiate the inverse hyperbolic sine function, we let

$$y = \sinh^{-1} u \tag{23}$$

or

$$u = \sinh y \tag{24}$$

Differentiating (24),

$$\frac{du}{dx} = \cosh y \frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{1}{\cosh y} \frac{du}{dx} \tag{25}$$

But $\cosh^2 y - \sinh^2 y = 1$, or $\cosh y = (1 + \sinh^2 y)^{1/2}$, so that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \sinh^2 y}} \frac{du}{dx} \tag{26}$$

Substituting (23) and (24) in (26),

$$\frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx} \tag{27}$$

Following a parallel course, we can show that

$$\frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx} \tag{28}$$

To differentiate

$$y = \tanh^{-1} u \tag{29}$$

we write

$$u = \tanh y \tag{30}$$

which differentiates to give

$$\frac{du}{dx} = \operatorname{sech}^2 y \frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} \frac{du}{dx} \tag{31}$$

The identity

$$1 - \tanh^2 y = \operatorname{sech}^2 y \tag{32}$$

makes (31) read

$$\frac{dy}{dx} = \frac{1}{1 - \tanh^2 y} \frac{du}{dx} \tag{33}$$

Substituting (29) and (30) in (33),

$$\frac{d}{dx} \tanh^{-1} u = \frac{1}{1 - u^2} \frac{du}{dx} \tag{34}$$

A similar procedure involving the function $y = \coth^{-1} u$, using the identity

$$\coth^2 y - 1 = \operatorname{csch}^2 y \tag{35}$$

yields

$$\frac{d}{dx} \coth^{-1} u = \frac{1}{1 - u^2} \frac{du}{dx} \tag{36}$$

If we let

$$y = \operatorname{sech}^{-1} u \tag{37}$$

or

$$u = \operatorname{sech} y \tag{38}$$

we get

$$\frac{du}{dx} = - \operatorname{sech} y \tanh y \frac{dy}{dx}$$

or

$$\frac{dy}{dx} = - \frac{1}{\operatorname{sech} y \tanh y} \frac{du}{dx} \tag{39}$$

The identity (32) makes (39) read

$$\frac{dy}{dx} = - \frac{1}{\operatorname{sech} y \sqrt{1 - \operatorname{sech}^2 y}} \frac{du}{dx} \tag{40}$$

Substituting (37) and (38) in (40),

$$\frac{d}{dx} \operatorname{sech}^{-1} u = - \frac{1}{u \sqrt{1 - u^2}} \frac{du}{dx} \tag{41}$$

A similar procedure involving the function $y = \operatorname{csch}^{-1} u$, using the identity (35) yields

$$\frac{d}{dx} \operatorname{csch}^{-1} u = - \frac{1}{u \sqrt{1 + u^2}} \frac{du}{dx} \tag{42}$$

From Fig. 13-5 we see that for negative values of u the positive root must be chosen, while for positive values of u the negative root is called for. Thus we rewrite (42)

$$\begin{aligned} \frac{d}{dx} \operatorname{csch}^{-1} u &= \frac{1}{u \sqrt{1 + u^2}} \frac{du}{dx} \qquad u < 0 \\ \frac{d}{dx} \operatorname{csch}^{-1} u &= - \frac{1}{u \sqrt{1 + u^2}} \frac{du}{dx} \qquad u > 0 \end{aligned} \tag{43}$$

QUESTIONS

1. Give formulas for differentiating (*a*) $\sinh^{-1} u$; (*b*) $\cosh^{-1} u$, (*c*) $\tanh^{-1} u$, (*d*) $\coth^{-1} u$, (*e*) $\operatorname{sech}^{-1} u$, (*f*) $\operatorname{csch}^{-1} u$.

2. Which values of the function $\cosh^{-1} u$ are included in the range of *principal values* of that function?

3. Same as question 2, except for the function $\operatorname{sech}^{-1} u$.

4. Compare each of the differentiation formulas of question 1 with the formula for differentiating the corresponding inverse circular function.

PROBLEMS

In Probs. 1 to 10 differentiate with respect to x.

1. $y = \sinh^{-1}(x/3)$
2. $y = \cosh^{-1} 4x$
3. $y = \tanh^{-1} 3x$
4. $y = \coth^{-1}(5x - 5)$
5. $y = \tanh^{-1} x^3$
6. $y = \operatorname{sech}^{-1} 10x$
7. $y = \operatorname{csch}^{-1} x^2$
8. $y = (\sinh^{-1} 2x)^2$
9. $y = x^2 \sinh^{-1} x$
10. $y = e^x \cosh^{-1} x$

11. The characteristic impedance of a certain transmission line is $Z_0 = \cosh^{-1}(b/c)$, where b is the center-to-center spacing of the conductors and c is their diameter. If $b = 1.2$ and $c = 1$, find the approximate change in Z_0 resulting from an increase in b to a new value of 1.21.

12. The *attenuation constant* of an m-derived low-pass filter section at its "infinite-attenuation" frequency is given by $\alpha = \sinh^{-1}[2m^2Q/(1 - m^2)]$, where Q is the figure of merit of the shunt inductor of the filter and m is a constant. Find $d\alpha/dQ$.

13. At a frequency f the *attenuation constant* of a high-pass constant-K filter section is $\alpha = 2\cosh^{-1}(f/f_c)$, where f_c is the cutoff frequency of the section. Find $d\alpha/df$.

14. The minimum loss in nepers for which a resistive pad can be designed, to match two impedances Z_1 and Z_2, is $N = \cosh^{-1}(Z_1/Z_2)^{1/2}$. If Z_2 is constant, find dN/dZ_1.

15. When the angular frequency ω of a signal applied to a constant-K low-pass filter is outside the passband, the attenuation constant becomes $\alpha = \cosh^{-1}[(\omega^2LC/2) - 1]$. At what rate does α vary with respect to ω?

16. A charged particle is released at a distance a from a stationary charge. The time t required for the freely moving particle to move a distance s due to the repulsion between the charges is $t = (1/c)(a/2)^{1/2}\{[s(s - a)]^{1/2} + a\cosh^{-1}(s/a)^{1/2}\}$ For constant values of a and c, find the speed at distance s.

13-8 Integrals yielding hyperbolic functions. We are now able to get integration formulas for several additional functions. From (12), we obtain

$$\sinh u\, du = d(\cosh u)$$

or

$$\int \sinh u\, du = \cosh u + C \tag{44}$$

In a similar way, (11) gives

$$\int \cosh u\, du = \sinh u + C \tag{45}$$

To integrate $\tanh u\, du$, write

$$\int \tanh u\, du = \int \frac{\sinh u\, du}{\cosh u}$$

Since the numerator in the right member is the differential of the denominator, the desired integral is

$$\int \tanh u\, du = \ln \cosh u + C \tag{46}$$

Similarly

$$\int \coth u\, du = \ln \sinh u + C \tag{47}$$

By (13),

$$\operatorname{sech}^2 u\, du = d(\tanh u)$$

so

$$\int \operatorname{sech}^2 u\, du = \tanh u + C \tag{48}$$

Similarly, from (14), we can get

$$\int \operatorname{csch}^2 u\, du = -\coth u + C \tag{49}$$

From (15) and (16), we quickly get the formulas

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C \tag{50}$$

and

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C \tag{51}$$

Notice how the integration formulas for hyperbolic functions compare, except for signs, with those for circular functions. The hyperbolic and inverse hyperbolic functions, including those of complex functions, are treated further in later parts of this book and in texts devoted to these functions.[3]

13-9 The Gudermannian. The function

$$\phi = \tan^{-1} (\sinh x)$$

is called the *Gudermannian* of x. It is denoted* by gd x. That is,

$$\operatorname{gd} x = \phi = \tan^{-1} (\sinh x) \tag{52}$$

Figure 13-6 is a graph of the Gudermannian function.

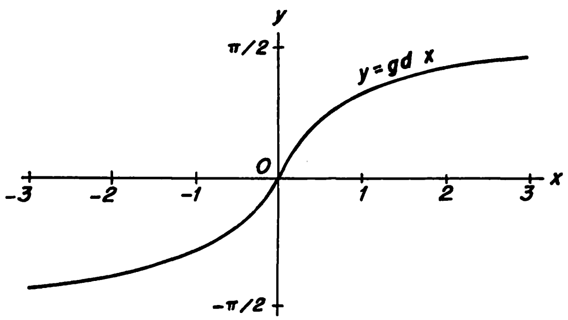


Fig. 13-6

* The Gudermannian of x is also sometimes called the *hyperbolic amplitude* of x (amh x or $\mathfrak{Amp}\ x$). Tables of the Gudermannian function appear on p. 500 of Griffin, "Mathematical Analysis—Higher Course" (ref. 5 of Chap. 1 of this book) and in Jahnke and Emde, "Tables of Functions" (ref. 4 of this chapter).

If we solve (52) for x, we obtain the *inverse Gudermannian* of ϕ (denoted by $\text{gd}^{-1}\,\phi$):

$$\text{gd}^{-1}\,\phi = x = \sinh^{-1}(\tan\phi) \tag{53}$$

We can show that

$$\frac{d}{dx}\,\text{gd}\,u = \text{sech}\,u\,\frac{du}{dx} \tag{54}$$

and

$$\frac{d}{dx}\,\text{gd}^{-1}\,u = \sec u\,\frac{du}{dx} \tag{55}$$

[Derivation of (54) and (55) is left to you in Probs. 18 and 19.] From these two formulas we get

$$\int \text{sech}\,x\,dx = \text{gd}\,x + C \tag{56}$$

and

$$\int \sec\phi\,d\phi = \text{gd}^{-1}\,\phi + C \tag{57}$$

QUESTIONS

1. State formulas for integrals of the following functions: (*a*) $\sinh u$, (*b*) $\cosh u$, (*c*) $\tanh u$, (*d*) $\coth u$, (*e*) $\text{sech}^2\,u$, (*f*) $\text{csch}^2\,u$, (*g*) $\text{sech}\,u\tanh u$, (*h*) $\text{csch}\,u\coth u$.
2. State a formula for ϕ, the Gudermannian of x.
3. State a formula for x, the inverse Gudermannian of ϕ.

PROBLEMS

In Probs. 1 to 10 integrate and check by differentiation.

1. $y = \int \sinh 2x\,dx$
2. $y = \int \sinh(x/5)\,dx$
3. $y = \int \cosh 32t\,dt$
4. $y = \int \tanh 9x\,dx$
5. $y = \int \coth a^2t\,dt$
6. $y = \int \text{sech}^2\,3\theta\,d\theta$
7. $y = \int \text{csch}^2\,120u\,du$
8. $y = \int \text{sech}\,x\tanh x\,dx$
9. $y = \int \text{csch}\,11t\coth 11t\,dt$
10. $y = \int x^2\cosh x^3\,dx$

11. Neglecting the effects of resistance, find the equation for the current required in a 5-henry inductor for the induced emf to be $v_{ind} = 12.4\cosh 20t$. Let $i = 0$ when $t = 0$.

12. A charging current $i_C = 0.05\sinh 200t$ was supplied to an initially discharged capacitor. What charge did this represent over a period of 1 millisecond?

13. The height (h feet) of an antenna wire over a flat earth at a horizontal distance x feet from the center point is $h = 200\cosh 0.005x - 60$. If it is suspended between towers 200 feet apart, find the area described by the wire, the towers, and the earth.

14. A current $i = 0.01\tanh 5{,}000t$ was supplied to an RC integrating circuit using a capacitor of 0.005 microfarad. Assuming that the capacitor was initially discharged, find an equation for the output voltage of the integrator.

15. Two coils have inductances $L_1 = 80$ millihenrys and $L_2 = 20$ millihenrys, respectively. The coefficient of coupling between them is $k = 0.1$. Find the current i_1 which must be supplied to L_1 for the induced emf in L_2 to be $v_2 = -2\sinh 0.1t$.

16. During a certain interval the current delivered to an originally discharged 2-microfarad capacitor was $i = 2\tanh 2{,}000t$, where t was in seconds. Find the capacitor voltage after 0.705 millisecond.

17. The current in a 100-ohm resistor is $i = 0.1 \text{ sech } 4t$. Find the average power dissipated in the resistor over a period from $t = 0$ to $t = \frac{1}{2}$ second.

18. Differentiate (52) to derive (54).

19. Differentiate (53) to derive (55).

20. Show that the angle at which a suspended cable is inclined at any point along the cable is equal to gd (x/a) radians if the height of the cable varies according to the catenary formula (10).

REFERENCES

1. HARRY W. REDDICK: "Differential Equations," 2d ed., pp. 9–12, John Wiley & Sons, Inc., New York, 1949.
2. C. R. WYLIE: "Calculus," pp. 340–344, McGraw-Hill BookCompany, Inc., New York, 1955.
3. A. E. KENNELLY: "Applications of Hyperbolic Functions to Electrical Engineering Problems," 3d ed., McGraw-Hill Book Company, Inc., New York, 1925.
4. E. JAHNKE and F. EMDE: "Tables of Functions," 4th ed., pp. 59–60 of Addenda, Dover Publications, New York, 1945.

Part Four

FURTHER OPERATIONS

14

Partial Derivatives

Thus far we have been studying the way in which a dependent variable changes with respect to a *single* independent variable. We now take up some cases where there are two or more independent variables.

14-1 Functions of two variables. Figure 14-1a shows the way in which the current i in a circuit varies as the resistance r is changed, assuming that the voltage v is kept constant at 10 volts. The graph of Fig. 14-1b is very similar, except that here the voltage has been changed to 20 volts; and in Fig. 14-1c is shown the current function when $v = 30$ volts.

If we should cut out these three graphs with scissors and space them at equal intervals indicating steps of 10 volts, the graphs would outline a *solid* figure, a "three-dimensional graph," as shown in Fig. 14-2. Imagine many additional graphs showing the behavior of i for *various* voltages between 0 and 10 volts, between 10 and 20 volts, and between 20 and 30 volts to be placed at appropriate intervals in this figure. The assemblage of such graphs can be thought of as defining an upper *surface* having, in this case, somewhat the shape of a tent.

The significance of Fig. 14-2 is this. Any point P upon the "floor" plane which includes the r and v axes defines a specific value of voltage

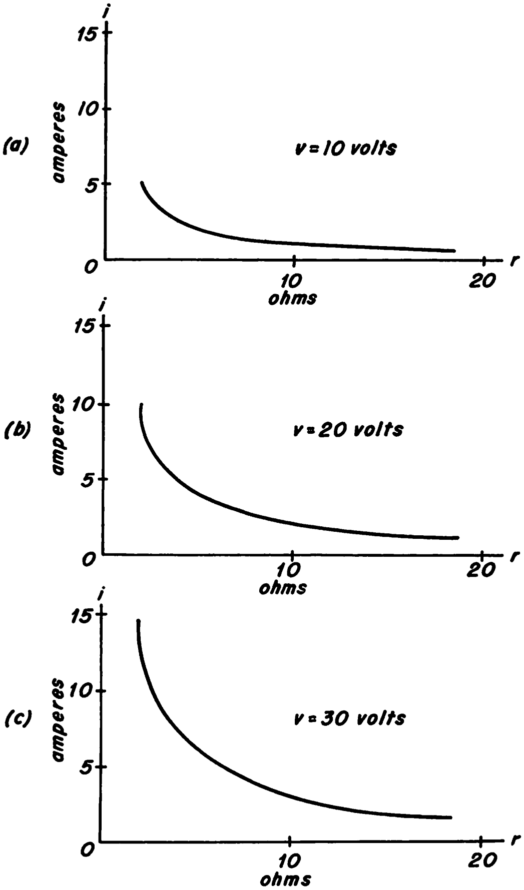


Fig. 14-1

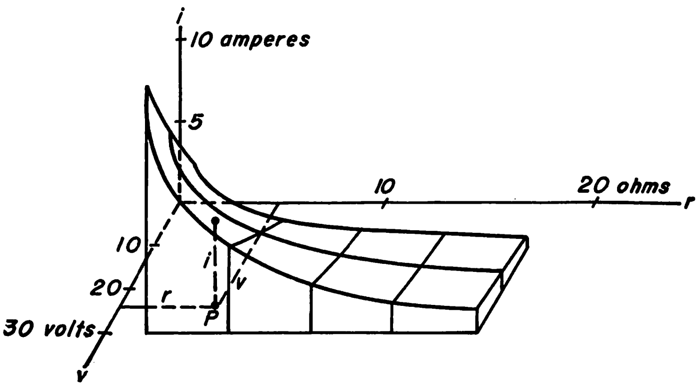


Fig. 14-2

(indicated by the distance of P from the r axis) and a particular value of resistance (indicated by the distance of P and the v axis). And the *height of the surface* directly above P indicates the current i which will flow in the circuit for these values of r and v. In other words, this solid graph is a pictorial representation of Ohm's law, and the equation for the surface height i is

$$i = \frac{v}{r} \tag{1}$$

(Note that it is the height of the *surface* of the figure which depicts the current function.)

14-2 Coordinate systems. *a. Cartesian coordinates.* For general

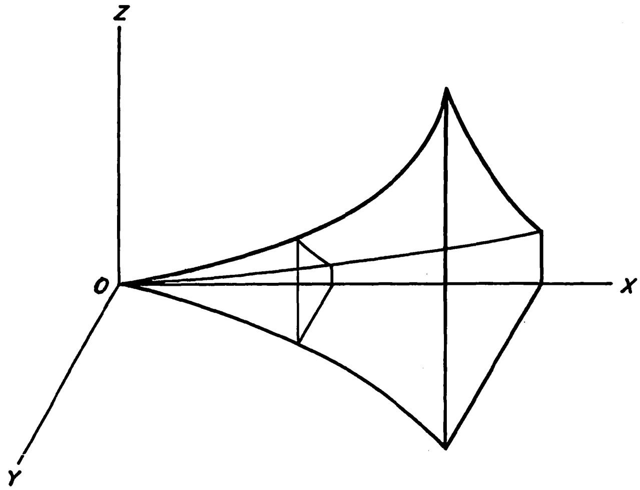


Fig. 14-3

discussions involving functions of two variables, we may consider a set of *cartesian coordinate axes* X, Y, and Z, as shown in Fig. 14-3. A dependent variable z, indicated by the height of the surface of the graph, is shown to vary as a function of two independent variables x and y. Here, we might say that "z is equal to the f function of x and y," written

$$z = f(x,y) \tag{2}$$

(In some texts, the designations of the x, y and z axes are interchanged with those shown here.)

b. Cylindrical coordinates. For some purposes it may be preferable to display a function of two independent variables in *cylindrical coordinates*, as in Fig. 14-4. Here the height z of a surface varies above points on the floor plane defined in terms of (a) the length r of a radius vector

and (b) the angle θ between the radius vector and a reference axis OX. In this system z is a function of r and of θ:

$$z = f(r,\theta) \tag{3}$$

c. Spherical coordinates. A third method of presenting a function of two independent variables is that of Fig. 14-5. Each point on the surface is expressed in terms of the length r of a radius vector, extending from a reference point or pole O to the point in question. The length r varies as a function of the *horizontal* (or *azimuth*) angle θ, by which the radius

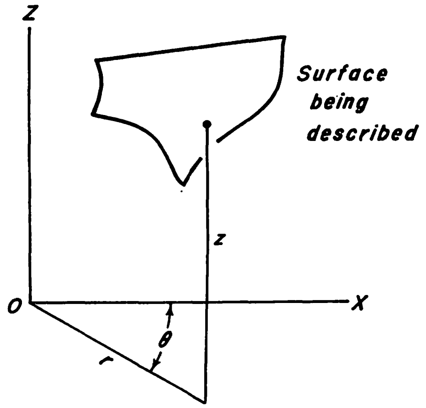


Fig. 14-4

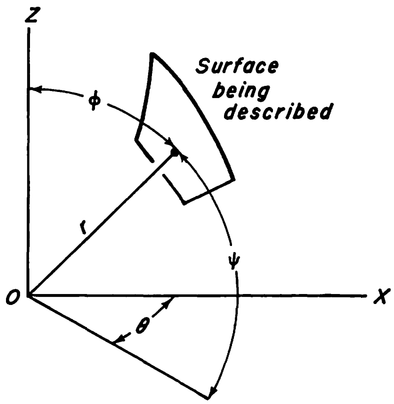


Fig. 14-5

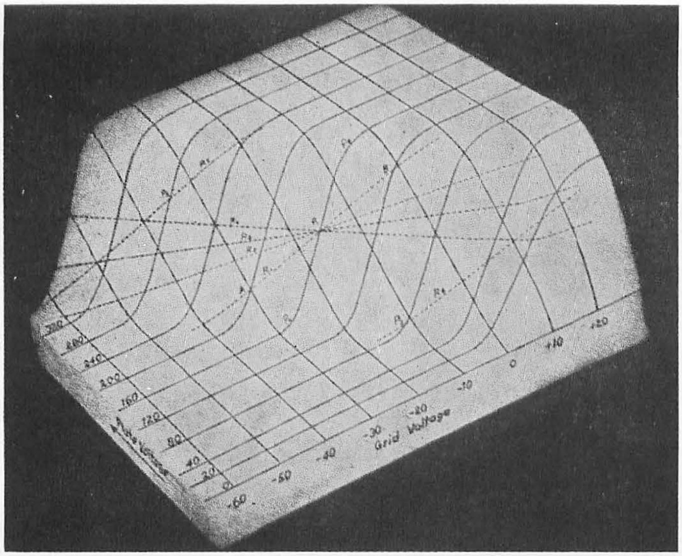
vector is displaced from a horizontal reference axis OX, and also as a function of a *vertical* angle ϕ by which the radius vector is displaced from a vertical reference axis OZ. That is, r is a function of θ and of ϕ:

$$r = f(\theta,\phi) \tag{4}$$

In some cases, it is convenient to consider, instead of the angle ϕ, an *elevation* angle ψ, by which the radius vector is displaced from the horizontal, as shown.

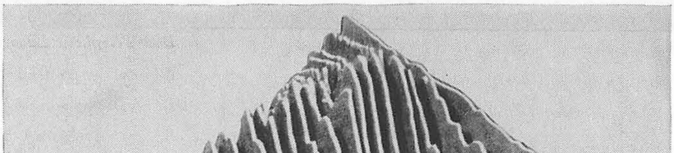
For the present, we shall limit ourselves principally to functions given in cartesian coordinates, that is, functions of the form $z = f(x,y)$.

14-3 Applications. Some applications of the preceding ideas will now be illustrated. The surface in Fig. 14-6a shows how the plate current (dependent variable) of a certain triode varies as a function of the plate voltage and the grid voltage applied to the tube. In Fig. 14-6b we see several *slices* of a solid graph, and these slices can be considered to define a surface showing how the impulse voltages in a helical coil varied with time and with distance. In Fig. 14-6c there is shown the manner in which the intensity of sound waves varied at different frequencies and at various times as the word "five" was spoken.



Austin V. Eastman, "Fundamentals of Vacuum Tubes," McGraw-Hill.

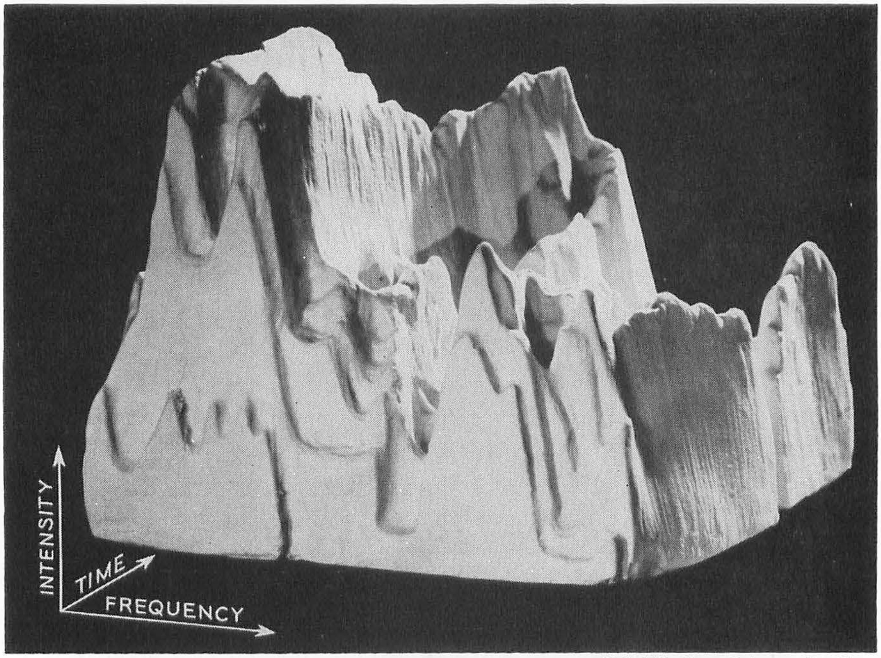
(a)



General Electric Co.

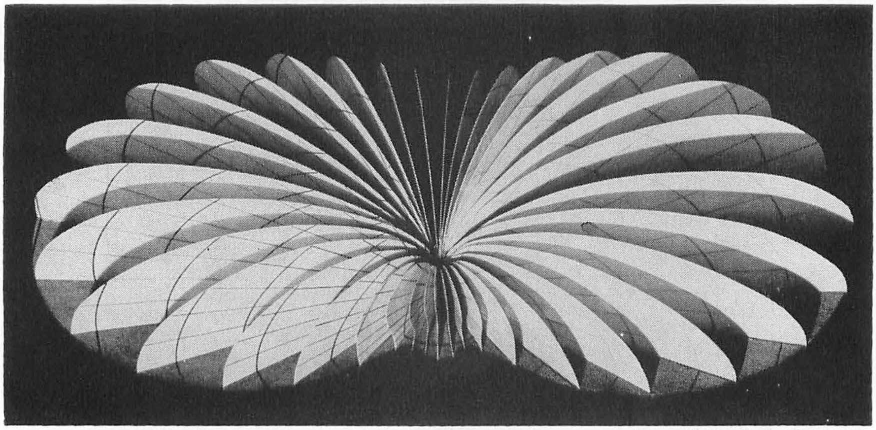
(b)

Fig. 14-6



Bell Telephone Laboratories, Inc.

(c)



James D. Parker, CBS Radio

(d)

Fig. 14-6 (*Continued*)

A graph in spherical coordinates is shown in Fig. 14-6*d*. The length of a radius vector, extending from the origin to a surface defined by the various *slices* shown, indicates the intensity of an unattenuated rf field at a distance of 1 mile from a certain directional antenna, varying as a function of angles in horizontal and in vertical planes.

No doubt you can think of other practical applications of solid graphs.

When a function varies with *three or more* independent variables, the physical world of our senses (a three-dimensional world) is insufficient for us to compose a solid graph of such a function. (But a few students seem to have a special gift for forming mental pictures of the way in which such functions might vary.) A special case is where a function varies with *time* and with two other independent variables, and we might try to imagine how the solid graph of such a function would be distorted as time goes on.

14-4 Partial derivatives. What kind of curve is cut from the solid graph

$$i = \frac{v}{r} \tag{1}$$

of Fig. 14-2 if we make a vertical cut from left to right where $v = 15$ volts (Fig. 14-7)? What is the slope of this curve where $r = 11$ ohms?

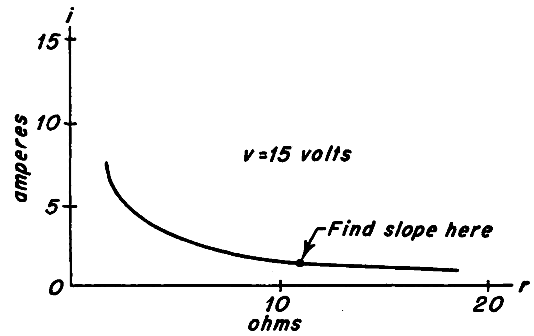


Fig. 14-7

To answer these questions, we let $v = 15$ in (1), getting

$$i = \frac{15}{r} \tag{5}$$

as the equation of the curve resulting from the cut. And this is easily differentiated to show that the slope of this curve is equal to $-15/r^2$. When $r = 11$, this becomes -0.124 ampere per ohm.

Now, whenever we differentiate a function of two variables, treating one of the variables as if it were a constant, we say that we are getting *the partial derivative of the function with respect to the other variable.* Here

we have calculated the partial derivative of i with respect to r, written $\partial i/\partial r$.

The symbol ∂ is a form of the letter d, and is called "round d" or "curly d."*

To illustrate the procedure used in practice, we again calculate the slope, in the direction of the r axis, of the section of Fig. 14-7, where $v = 15$ volts. We are given that

$$i = \frac{v}{r} \tag{1}$$

Treating v as a constant and differentiating with respect to r,

$$\frac{\partial i}{\partial r} = -\frac{v}{r^2} \tag{6}$$

When $v = 15$ volts, this becomes

$$\frac{\partial i}{\partial r} = -\frac{15}{r^2} \tag{7}$$

If we want the slope of the curve of Fig. 14-7, where $r = 11$ ohms, we simply let $r = 11$ in (7), getting $\partial i/\partial r = -0.124$.

Conversationally, we might say that "the current varies with resistance at a rate of minus 0.124 ampere per volt, *other things being equal.*"

Example 1. The height z of a surface varies with x and y according to $z = x^2 + 2xy + y^2$. (*a*) Find a formula for $\partial z/\partial y$. (*b*) Find $\partial z/\partial y$ when $x = 2$ and $y = 3$.

Treating x as a constant and differentiating with respect to y, we get as the answer to part (*a*):

$$\frac{\partial z}{\partial y} = 2(x + y)$$

Letting $x = 2$ and $y = 3$, we obtain for part (*b*):

$$\frac{\partial z}{\partial y} = 2(2 + 3) = 10$$

Thus, at the point where $x = 2$ and $y = 3$, z is increasing at the rate of 10 units for each unit of y.

Example 2. If a function v varies with w, x, and y as $v = w^2 + w \cos xy + e^y$, find (*a*) $\partial v/\partial w$; (*b*) $\partial v/\partial x$; (*c*) $\partial v/\partial y$.

* In conversation, the term *partial derivative* is sometimes abbreviated to simply *partial.* We might read $\partial i/\partial r$ variously as "the partial derivative of i with respect to r," or "the partial of i with respect to r," or "round d i over round d r," etc. Still another spoken form is "die i over die r."

PROBLEMS

1. If $z = 2x^2y + y^2$, find $\partial z/\partial y$.
2. If $z = 10xy - y^3 - 3y^2$, find $\partial z/\partial x$.
3. If $z = x^2 - y^2 - xy$, find $\partial z/\partial x$.
4. If $q = 2rt^2 - (rt)^{1/2} + r^2$, find $\partial q/\partial r$.
5. If $w = 15u^2 - 22u + 8uv^3$, find $\partial w/\partial u$.
6. If $z = ax^ny^n - bx + cxy^m$, find $\partial z/\partial y$.
7. If $i = t^3 + 3Tt^2$, find $\partial i/\partial t$.
8. If $z = xe^y + e^x$, find $\partial z/\partial x$.
9. If $v = e^x \cos\theta - e^x \sin x\theta$, find $\partial v/\partial \theta$.
10. If $v = t^2 \sin t\theta + \theta \sin t$, find $\partial v/\partial t$.
11. If $z = y^2e^{xy} \sin y + e^y \cos xy$, find $\partial z/\partial y$.
12. If $z = x \sin^2 xy - e^x \cos xy$, find $\partial z/\partial x$.
13. If $z = \sin^{-1} xy + 2y^2$, find $\partial z/\partial y$.
14. If $q = \cosh x^2 + 2 \sinh xy$, find $\partial q/\partial x$.
15. If $r = \sin xy \cosh y + 2 \sinh xy$, find $\partial r/\partial y$.

16. The output power available from a source of emf V having an internal resistance R is $p_{\max} = V^2/4R$. Find the rate at which $p_{\max}$ varies (a) as a function of V and (b) as a function of R.

17. An equation expressing the intensity of a sound wave is $I = 2\pi^2cf^2A^2\rho$. Find the rate at which the intensity changes (a) with respect to f and (b) with respect to ρ.

18. The coefficient of coupling between two coils of inductances L_1 and L_2 is $k = M/(L_1L_2)^{1/2}$, where M is the mutual inductance between the coils. Find at what rate k varies (a) with respect to M and (b) with respect to L_1.

19. The thermal noise emf generated in a resistance R ohms at an absolute temperature T degrees, over a bandwidth B cycles, is $v_N = 2(kTBR)^{1/2}$, where k is a constant. Find $\partial v_N/\partial R$.

20. A light source of I candlepower located r feet from a surface inclined at an angle θ provides an illumination $E = (I \cos\theta)/r^2$ foot-candles. Find the rate at which E varies (a) with respect to I and (b) with respect to θ.

21. The torque $\mathbf{T}$ acting upon a single turn of wire in a magnetic field of intensity $\mathbf{B}$, when the wire carries a current I, is $\mathbf{T} = 2\mathbf{B}lIr \cos\theta$, where l is the length of the wire in a direction perpendicular to the field, θ is the angle at which the coil is inclined to the field, and $2r$ is the width of the coil. Find the rate at which $\mathbf{T}$ changes (a) with respect to $\mathbf{B}$ and (b) with respect to θ.

22. From the Ohm's-law formula $IR = V$, show: (a) $\partial I/\partial V = 1/R$; ($b$) $\partial V/\partial R = I$, ($c$) $\partial I/\partial R = -V/R^2$ $(= -I/R)$, (d) for *partial* derivatives $(\partial I/\partial V)(\partial V/\partial R)$ does *not* equal $\partial I/\partial R$, as we should expect for *ordinary* derivatives.

23. In a series ac circuit, $Z^2 = R^2 + X^2$. And $R = Z \cos\theta$ (where θ is the impedance phase angle). The first equation gives $\partial Z/\partial R = R/Z$ $(= \cos\theta)$; the second gives $\partial R/\partial Z = \cos\theta$. Show that for *partial* derivatives $\partial Z/\partial R \neq 1/(\partial R/\partial Z)$, as contrasted with the case for *ordinary* derivatives.

14-5 Partial and total differentials; total derivatives. Let $z = f(x,y)$. Suppose y to be held constant, and let $\Delta_x z$ be the change in z resulting from a change Δx in x. Then

$$\Delta_x z \approx \frac{\partial z}{\partial x}\,\Delta x \qquad \text{if } \Delta x \text{ is very small} \tag{12}$$

You can readily confirm these results:

(a) $$\frac{\partial v}{\partial w} = 2w + \cos xy$$

(b) $$\frac{\partial v}{\partial x} = -wy \sin xy$$

(c) $$\frac{\partial v}{\partial y} = -wx \sin xy + e^y$$

It should be pointed out that, in general, we do not treat a partial derivative (say $\partial z/\partial x$) as if it were a fraction made up of ∂z and ∂x, for the expression ∂z, for instance, does not alone indicate which quantities are held constant and which permitted to vary.

An application of partial derivatives is in electron-tube theory. In a triode, for example, we may be interested in the plate current i_b, plate voltage v_b, and grid voltage v_c. Since there are more than two variables, the use of partial derivatives properly describes the tube characteristics:

$$g_m = \frac{\partial i_b}{\partial v_c} \quad \text{mhos} \tag{8}$$

$$r_p = \frac{\partial v_b}{\partial i_b} \quad \text{ohms} \tag{9}$$

$$\mu = -\frac{\partial v_b}{\partial v_c} \tag{10}$$

Similarly, the current gain α of a transistor may properly be expressed

$$\alpha = \frac{\partial i_c}{\partial i_e} \tag{11}$$

where i_c and i_e are collector and emitter currents, respectively.

To emphasize that a certain independent variable (say y) or variables (say y and u) are held constant as we take the partial derivative of z with respect to the independent variable x, we may write

$$\left.\frac{\partial z}{\partial x}\right]_y \qquad \text{or} \qquad \left.\frac{\partial z}{\partial x}\right]_{y,u}$$

QUESTIONS

1. Describe how a function of two variables may be displayed by means of a solid graph in (a) cartesian coordinates, (b) cylindrical coordinates, (c) spherical coordinates.
2. If $z = f(x,y)$, and if we differentiate z with respect to y, treating x as if it were a constant, what name is applied to the result?
3. Give a name for the symbol ∂.
4. Express the following electron-tube and transistor characteristics in their partial-derivative forms: (a) g_m, (b) r_p, (c) μ, (d) α.

Compare (12) with a similar equation [Eq. (6) of Sec. 6-3] involving a function y of a single variable x:

$$\Delta y \approx f'(x)\, \Delta x \qquad \text{if } \Delta x \text{ is very small}$$

Instead of the latter form, we recall that Eq. (3) of Sec. 6-4 is often used symbolically, to avoid reverting to the delta notation:

$$dy = f'(x)\, dx$$

In a parallel manner we may express, from (12) above, what we shall call the x *partial differential of* z, represented by $d_x z$ (or $\partial_x z$):

$$d_x z = \frac{\partial z}{\partial x}\, dx \tag{13}$$

The form (13) is often used to avoid the delta notation in expressing approximately the change $d_x z$ in z resulting from a small increment dx in x. (As mentioned in Chap. 6, regarding functions of a single variable, such usage of the differential notation is loose but not uncommon.) Note that (13) gives *precisely* the change which would take place in z if the exact rate $\partial z/\partial x$ were maintained throughout the interval under discussion.

Other partial differentials are similarly arrived at. The y partial differential of z, for instance, is given by

$$d_y z = \frac{\partial z}{\partial y}\, dy$$

It expresses the approximate change in z resulting from a small change in y, with any other independent variables being held constant.

Example 1. The power in a circuit is $p = i^2 r$. (*a*) Find a formula for $d_i p$. (*b*) Find the approximate change in p if i changes from 10 to 10.1 amperes, r remaining constant at 2 ohms.

To solve part (*a*) we find $\partial p/\partial i = 2ir$. Then

$$d_i p = \frac{\partial p}{\partial i}\, di = 2ir\, di$$

And part (*b*) is found by letting $i = 10$, $r = 2$, and $di = 0.1$:

$$d_i p = 2(10)(2)(0.1) = 4 \text{ watts}$$

Now consider what happens to a function z if *all* the independent variables are permitted to change simultaneously. For instance, if z is a function of the two variables x and y, then the x partial differential and the y partial differential are

$$d_x z = \frac{\partial z}{\partial x}\, dx \qquad \text{and} \qquad d_y z = \frac{\partial z}{\partial y}\, dy$$

respectively. The *sum* of the various partial differentials is called the *total differential* of z, indicated by dz:

$$dz = d_x z + d_y z = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \tag{14}$$

with further partial differentials included in the sum if there are more than two independent variables. The total differential dz may be used to express approximately the effect upon z of small changes occurring simultaneously in all the independent variables, as illustrated in the following example.

Example 2. If $z = w^2 + wxy^2$, and if $w = 2$, $x = 1$, and $y = 2$, find the approximate change in z resulting from an increase of 0.1 in w, a decrease of 0.3 in x, and an increase of 0.1 in y.

We find $\partial z/\partial w = 2w + xy^2$; $\partial z/\partial x = wy^2$; $\partial z/\partial y = 2wxy$. Applying the concept of (14), we find that the approximate change in z is

$$dz = \frac{\partial z}{\partial w}\,dw + \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = (2w + xy^2)\,dw + wy^2\,dx + 2wxy\,dy$$

$$= (2 \times 2 + 1 \times 2^2)(0.1) + (2 \times 2^2)(-0.3) + (2 \times 2 \times 1 \times 2)(0.1) = -0.8$$

If we now let each of the independent variables in (14) vary simultaneously as functions of some variable (say t), we can get the *total derivative* of z by dividing each term of (14) by dt:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{15}$$

with further terms being provided, of course, if there are more than two independent variables. This gives the rate of change of z with respect to t, as all the independent variables x, y, etc., change simultaneously.*

QUESTIONS

1. If $z = f(x,y)$, what symbol is often used to indicate the approximate change in z resulting from a small change in x?
2. What is the name of the quantity represented by $d_y z$?
3. If $z = f(x,y)$, what name is given to the quantity represented by dz?
4. If $z = f(x,y)$, what symbol represents the approximate change in z resulting from simultaneous small changes in x and y?
5. If $z = f(x,y)$, give general formulas for (*a*) the x partial differential of z; (*b*) the y partial differential of z; (*c*) the total differential of z; (*d*) the total derivative of z.

PROBLEMS

1. When a current i flows in a resistance r, the power developed is $p = i^2r$. Find a formula for the r partial differential of p.

* The formulas presented in this section are developed formally in many standard calculus texts.

2. A voltage v applied across a resistance r results in a power dissipation $p = v^2/r$. Get an equation for $d_r p$.

3. A voltage drop $v = ir$ appears across a resistance r when a current i flows through it. When $i = 5$, what approximate change occurs in v as r changes from 10 to 10.2?

4. The reactance in ohms due to a capacitance C at a frequency f is $X_C = 1/2\pi fC$. Find the approximate change in X_C resulting from a change in f from 10^6 to 1.05×10^6 cycles when $C = 5 \times 10^{-10}$ farad.

5. The reactance offered at a frequency f cycles by an inductor of L henrys is $X_L = 2\pi fL$ ohms. Find a formula expressing the approximate total change in X_L due to small changes in f and L occurring simultaneously.

6. If the inductance of a certain solenoid is $L = KDN^2$, where K is a *shape factor*, D is the diameter in inches, and N is the number of turns, give an equation for the approximate change in L caused by simultaneous small changes in K, D, and N.

7. The energy stored in an inductor is $w = Li^2/2$. If L changes from 2 to 2.02 henrys, and if i simultaneously changes from 1 to 0.9 ampere, find the approximate change in w.

8. When a current i flows in a single-turn coil of radius r, the resulting magnetic field intensity at a point P on the axis of the coil is $\mathbf{H} = (2\pi i \sin^3 \theta)/r$, where θ is the angle formed between the coil axis and a line connecting P with a point on the coil circumference. If r is constant, and if i and θ are changed simultaneously, find a formula for $d\mathbf{H}/dt$.

9. A rectangular coil of length l and width $2r$ has N turns. If the coil is placed in a magnetic field of density $\mathbf{B}$ and a current i is sent through the coil, the resulting torque on the coil is $\mathbf{T} = 2\mathbf{B}liNr \cos \theta$, where θ is the angle between the plane of the coil and the direction of the flux. If l, N, and r are constants, and if $\mathbf{B}$, i, and θ are simultaneously varied, find a formula for $d\mathbf{T}/dt$.

10. The voltage amplification, under certain conditions, of a triode amplifier is $A_v = \mu R_L/(R_L + r_p)$. If R_L is changed, and if simultaneously the voltages applied to the tube elements are varied so as to affect μ and r_p, give a formula expressing the time rate of change of A_v.

14-6 Higher partial derivatives.

If z is a function of x and y, and if we hold y constant and get the partial derivative of z with respect to x, the result is, of course, $\partial z/\partial x$. If we differentiate $\partial z/\partial x$ with respect to x, the result is expressed as $\partial^2 z/\partial x^2$, called the "second partial derivative of z with respect to x."*

Similarly, a further differentiation gives the third partial derivative of z with respect to x, $\partial^3 z/\partial x^3$, and so on.

Example 1. The power in a circuit is $p = i^2 r$. Find $\partial^2 p/\partial i^2$.

The first partial derivative of p with respect to i is found as $\partial p/\partial i = 2ir$; differentiating this with respect to i gives $\partial^2 p/\partial i^2 = 2r$.

Again supposing that $z = f(x,y)$, let us now

1. Hold y constant and perform a first differentiation with respect to x. As we have seen, the result is called $\partial z/\partial x$.

* This can be read aloud in various ways, such as "the second partial of z with respect to x" or "die two z over die x squared," etc.

2. In the result $\partial z/\partial x$, hold x constant and perform a differentiation with respect to y. This result we shall call

$$\frac{\partial^2 z}{\partial y\,\partial x}$$

(note the *order* of the symbols). Similarly for further differentiations.*

Example 2. A cube whose edges are s units long and whose density is D has a mass $m = s^3D$. Find $\partial^2 m/\partial s\,\partial D$.

Differentiation with respect to D gives $\partial m/\partial D = s^3$. Then, differentiating with respect to s, we get $\partial^2 m/\partial s\,\partial D = 3s^2$.

In most functions ordinarily encountered in practice, the results are identical regardless of the order of the differentiation. But this is not always true (Sec. 14-9).

QUESTIONS

1. If $z = f(x,y)$, what name is applied to the result of differentiating z twice with respect to x?
2. What symbol is applied to the result of question 1?
3. If $z = f(x,y)$, what symbol signifies the result of differentiating z first with respect to x and then with respect to y?

PROBLEMS

1. A voltage v is applied to a resistance r ohms, so that the power in the resistance is $p = v^2/r$. Find $\partial^2 p/\partial v^2$.

2. From the formula for the energy stored in a capacitor, $w = Cv^2/2$, find $\partial^2 w/\partial v^2$.

3. A resistance R is connected in series with a reactance X, so that the impedance of the combination is $\boldsymbol{Z} = (R^2 + X^2)^{1/2}$. Find $\partial^2 \boldsymbol{Z}/\partial R^2$.

4. The intensity $\mathbf{E}$ of the electric field associated with a transmitted wave at a given instant varies with distance s from the antenna according to $\mathbf{E} = \mathbf{E}_{\max} \sin(2\pi s/\lambda)$, where $\mathbf{E}_{\max}$ is the crest value of the field and λ is the wavelength. Find $\partial^2\mathbf{E}/\partial s\,\partial\lambda$.

5. In Prob. 3, find $\partial^2\boldsymbol{Z}/\partial X\,\partial R$.

6. In Prob. 5, show that the result would have been unchanged had we reversed the order of the differentiation, taking $\partial^2\boldsymbol{Z}/\partial R\,\partial X$.

7. When a capacitor is discharged through a resistor, the capacitor voltage follows the equation $v_C = Ve^{-t/RC}$. Find $\partial^2 v_C/\partial R\,\partial t$, assuming V constant.

8. When a symmetrical T attenuator connects circuits whose impedance is $\boldsymbol{Z}$, each of the series arms of the attenuator has a resistance $R_1 = \boldsymbol{Z}\tanh(N/2)$, where N is the desired attenuation in nepers. Find $\partial^2 R_1/\partial\boldsymbol{Z}\,\partial N$.

9. The current in a certain antenna varied with distance s from one end of the antenna and with time t according to $i = I_0 \sin(2\pi s/\lambda)\sin 2\pi ft$, where I_0 was the peak current, λ the wavelength, and f the operating frequency. If I_0, λ, and f were constants, find $\partial^2 i/\partial s\,\partial t$.

* In some treatments, f_x represents the partial derivative of a function f with respect to x, and f_{yx} indicates the result of two partial differentiations, first with respect to x and then with respect to y.

10. When a steady voltage V is applied to an inductor of inductance L and resistance R, the current rises according to $i = (V/R)(1 - e^{-Rt/L})$. Find $\partial^2 i/\partial L\, \partial t$.

14-7 Maxima and minima. Let z be a function of x and y (Fig. 14-8). Let it be desired to find any points on the surface representing z at which z has maximum or minimum values. (For present purposes we limit ourselves to functions of two independent variables, but the discussion can readily be extended to cases in which further independent variables are involved.)

If z rises *smoothly* to a maximum at any point, that is, if the maximum is analogous to the ordinary maxima studied in Chap. 8, then *at the*

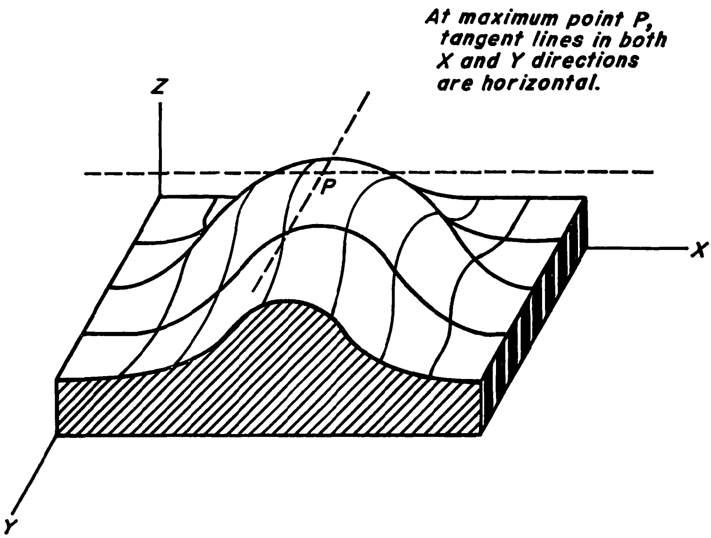


Fig. 14-8

maximum point we must have

$$\frac{\partial z}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 0 \tag{16}$$

so that tangent lines in the x and y directions will be horizontal, as shown in the figure. Similarly for a minimum point.

Conditions (16) are *necessary* for the existence of a maximum or minimum of the kind discussed here, but they are not *sufficient*. In other words, if the conditions (16) apply at a given point, then that point is a *possible* maximum or minimum point. Tests which establish definitely the existence of maxima or minima are rather complicated.[1,2] A partial check is to try values of x and of y on each side of the supposed maximum or minimum. In many practical problems, no check is required.

Example 1. Test for maxima or minima

$$z = x^2 + y^2 - 8x - 2y + 100 \tag{A}$$

We find

$$\frac{\partial z}{\partial x} = 2x - 8 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2y - 2 \tag{B}$$

Setting the derivatives in (B) equal to zero, we find

$$x = 4 \qquad \text{and} \qquad y = 1 \tag{C}$$

Thus, the point (4,1) satisfies (16). [Had *both* x and y appeared in either equation (B), these equations would have had to be solved *simultaneously*.] Now we test the point (4,1) by substituting x and y slightly smaller and larger than $x = 4$ and $y = 1$ in (B):

$$\text{When } x = 3.9 \text{ and } y = 1 \qquad \frac{\partial z}{\partial x} < 0$$

$$\text{When } x = 4.1 \text{ and } y = 1 \qquad \frac{\partial z}{\partial x} > 0$$

$$\text{When } y = 0.9 \text{ and } x = 4 \qquad \frac{\partial z}{\partial y} < 0$$

$$\text{When } y = 1.1 \text{ and } x = 4 \qquad \frac{\partial z}{\partial y} > 0$$

Thus the surface representing z appears to drop, then rise as we go past the point (4,1) in either the x or the y direction. Barring a more complete test, we assume that there exists a *minimum* at (4,1). If we wish to *evaluate* this minimum, we substitute (C) in (A):

$$z_{\min} = 16 + 1 - 32 - 2 + 100 = 83$$

Example 2. A parts cabinet is to contain 500 cubic inches and is to consist of a minimum amount of material. Find the dimensions if there is to be no top on the cabinet.

Letting x and y be the length and width and z the depth of the cabinet, we find that the volume is

$$xyz = 500$$

It is essential in all these problems that all independent variables used be actually independent. Here z may be expressed in terms of x and y:

$$z = \frac{500}{xy} \tag{A}$$

The material used in constructing the cabinet will then have a surface area

$$S = xy + 2\left(\frac{500}{x} + \frac{500}{y}\right)$$

We get

$$\frac{\partial S}{\partial x} = y - \frac{1{,}000}{x^2} \tag{B}$$

$$\frac{\partial S}{\partial y} = x - \frac{1{,}000}{y^2} \tag{C}$$

Setting the derivatives in (*B*) and (*C*) equal to zero,

$$y = \frac{1{,}000}{x^2} \tag{D}$$

$$x = \frac{1{,}000}{y^2} \tag{E}$$

Substituting (*D*) in (*E*),

$$x = \frac{x^4}{1{,}000} \tag{F}$$

Since $x \neq 0$, we divide (*F*) by x and solve:

$$x^3 = 1{,}000$$

$$x = 10 \tag{G}$$

Substituting (*G*) in (*D*),

$$y = 10 \tag{H}$$

Substituting (*G*) and (*H*) in (*A*),

$$z = 5$$

Thus the box should have length and width 10 inches and depth 5 inches.

14-8 Gradients. Let the temperature T of a hot metal plate vary from point to point along a line in the x direction at a rate $\partial T/\partial x$, and let T vary along a line in the y direction at a rate $\partial T/\partial y$. We may speak of $\partial T/\partial x$ and $\partial T/\partial y$ as the *gradients* of T in the x and y directions, respectively.

Let dT/ds express the rate of change of T with distance s along a line AB which makes a horizontal angle α with the x axis. Here x and y are functions of s, so that we may consider dT/ds as a total derivative [Eq. (15), where s replaces t]:

$$\frac{dT}{ds} = \frac{\partial T}{\partial x}\frac{dx}{ds} + \frac{\partial T}{\partial y}\frac{dy}{ds} \tag{17}$$

The quantity dT/ds is called the *directional gradient* (or *directional derivative*) of T in the direction of AB. Proceeding along AB, the value of dT/ds at any point (x_1,y_1) indicates the rate of change of temperature (say in degrees per inch) at that point.

A second expression for dT/ds may be derived. At any point (x,y) on AB,

$$x - x_1 = s \cos \alpha \qquad \text{and} \qquad y - y_1 = s \sin \alpha \tag{18}$$

For the time being, consider the point (x_1,y_1) as fixed, and further, let α be constant. Differentiating with respect to s in (18),

$$\frac{dx}{ds} = \cos \alpha \qquad \text{and} \qquad \frac{dy}{ds} = \sin \alpha \tag{19}$$

Substituting (19) in (17), we get for the directional gradient

$$\frac{dT}{ds} = \frac{\partial T}{\partial x} \cos \alpha + \frac{\partial T}{\partial y} \sin \alpha \tag{20}$$

Expressions like (20) can be used for directional gradients, not only of temperature, but of other quantities like magnetic and electric potential, density of a solid, and velocity of a fluid.

Example 1. If the temperature varied over a flat plate according to $T = 10 + 4x + 6\sqrt{3}y - x^2 - y^2$, find the directional gradient of T in the direction $\alpha = 60°$, at the point $(1,\sqrt{3})$.

We find

$$\frac{\partial T}{\partial x} = 4 - 2x \qquad \text{and} \qquad \frac{\partial T}{\partial y} = 6\sqrt{3} - 2y \tag{A}$$

Substituting (A) in (20),

$$\frac{dT}{ds} = (4 - 2x)\cos \alpha + (6\sqrt{3} - 2y)\sin \alpha \tag{B}$$

Letting $x = 1$, $y = \sqrt{3}$,

$$\frac{dT}{ds} = 2\cos \alpha + 4\sqrt{3}\sin \alpha \tag{C}$$

When $\alpha = 60°$, (C) becomes

$$\frac{dT}{ds} = 1 + 6 = 7 \text{ degrees per unit distance}$$

The *maximum* value assumed by dT/ds as α is varied, if such a maximum occurs, is called the *maximum gradient* (or simply the *gradient*) of T.

Example 2. Find (a) the direction and (b) the value of the maximum gradient of the temperature function of Example 1, at $(1,\sqrt{3})$.

Differentiating (C) of Example 1, with respect to α, and setting the result equal to zero,

$$\frac{d}{d\alpha}\left(\frac{dT}{ds}\right) = -2\sin \alpha + 4\sqrt{3}\cos \alpha = 0$$

$$\sin \alpha = 2\sqrt{3}\cos \alpha$$

$$\tan \alpha = 2\sqrt{3} = 3.4641$$

$$\alpha = 73.9° \qquad \text{(from tables)}$$

Thus 73.9° is a possible answer to part (*a*) of our problem. Testing with values of α near to 73.9° (say 72 and 76°) we find that dT/ds has a derivative which increases, then decreases, as α goes through 73.9°. Thus there is a maximum gradient in that direction. To find the value of this maximum gradient we let $\alpha = 73.9°$ in (*C*) of Example 1:

$$\left(\frac{dT}{ds}\right)_{\max} = 2 \cos 73.9° + 4 \sqrt{3} \sin 73.9° = 7.21$$

QUESTIONS

1. If $z = f(x,y)$, what are two necessary conditions for the existence of a maximum or minimum value of z at any given point (assuming that z does not rise *sharply* to a maximum or fall sharply to a minimum)?

2. Are the two necessary conditions required in question 1 *sufficient* for the existence of a maximum or minimum of the kind intended in that question?

3. If $z = f(x,y)$, what is the name for the quantity which expresses the rate of change of z with respect to distance s along a given straight line?

4. Give two forms of a formula for the quantity required in question 3.

5. What term is applied to the greatest value of the quantity required in question 3?

PROBLEMS

In Probs. 1 to 5 test for maxima and minima.

1. $z = x^2 + y^2 - 4y + 3$

2. $z = x^2 + y^2 + 6x - 10y + 8$

3. $z = e^{-x^2} - y^2 + 2y + 1$

4. $D = x^2 + 2x^3 + 4y$

5. $z = 10 - x^2 - y^2 + x^2y^2$

6. If the temperature varied over a flat plate according to $T = e^x \sin y$, find the directional gradient of T in the direction $\alpha = 60°$, at the point $(0,\pi/2)$.

7. If the density D of an insulating sheet varies as $D = x^2 + 2x^3 + 4y$, find the directional gradient of D in the direction $\alpha = 45°$ at the point (1,1).

8. In Prob. 7 find the angle at which the maximum gradient occurs at (1,1).

9. A metal sheet formed a part of the anode of a transmitting tube. The temperature varied thus over the sheet: $T = 300(3 \operatorname{sech} x + 5e^{-y^2})$. Show that the maximum temperature occurred at the point (0,0).

10. Obtain a formula for the temperature gradient in any direction α for the sheet of Prob. 9 at the point (1,1).

11. A rectangular parts box is to have a volume of 1,000 cubic inches. Prove that the least material is required for the box, including cover, if the box is a cube having an edge of 10 inches.

12. A charged wire of radius r is located vertically along the z axis. A thin, flat metal plate is located at a distance x_1 from the wire and parallel to the wire and to the y axis. The plate also carries a charge. It can be shown that, under appropriate conditions, the *electric potential* at any point $P(x,y)$ between the plate and the wire is $V = K - K_1(x_1 - x) - K_2 \ln (\sqrt{x^2 + y^2}/r)$, where K, K_1, and K_2 are constants. Find a formula for the directional gradient of potential in a direction α.

13. Table 14-1 shows how the plate current i_b in milliamperes of a triode varied with plate voltage v_b. It was desired to approximate this table by a single *linear* equation of the form $i_b = Av_b + B$ (where A and B are constants). Determine the

best values for A and B, establishing the desired formula. (HINT: It is common to take the "best" values of A and B as those which minimize the *sum E of the squares of all differences between the graph values and the table values.*)

Table 14-1

v_b..........	1,600	1,800	2,000	2,200	2,400
i_b..........	0	20	100	200	380

14-9 Conclusion. It has been mentioned that we do not take a partial derivative $\partial z/\partial x$ as a fraction involving ∂z and ∂x. As a consequence, certain features of ordinary derivatives like dy/dx do not appear in partial derivatives. For example, in Prob. 37 following Sec. 6-1 it was pointed out that

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

We must not make the mistake of assuming that a similar relation necessarily applies to partial derivatives like $\partial z/\partial x$ and $\partial x/\partial z$ (Sec. 14-4, Probs. 22 and 23).

In getting higher partial derivatives (Sec. 14-6) the order in which the differentiations are carried out usually does not alter the result. It can be proved that this is true for any case wherein the successive derivatives are continuous functions of all the variables involved.

REFERENCES

1. C. R. WYLIE: "Calculus," pp. 395–399, McGraw-Hill Book Company, Inc., New York, 1953.
2. H. M. BACON: "Differential and Integral Calculus," 2d ed., pp. 381–385, McGraw-Hill Book Company, Inc., New York, 1955.

15

Integration Techniques

Formulas have been presented in previous chapters for obtaining the integrals of a variety of functions, but they are not sufficient for all cases. Here we consider procedures for integrating further functions.

15-1 Introduction. No general rules can be given for obtaining the integral of any function which might be written down at random. Success in integrating depends upon such factors as these:

1. Familiarity with the basic formulas which have heretofore been obtained for integrating many simple type forms of functions. These formulas are condensed, for ready reference and review, in Table 11 on the inside back cover of this book.

2. Availability of tables of integrals of various other type forms of not so simple a nature. These formulas have been obtained by mathematicians in various ways, some of them quite advanced. A brief table is given as Table 9 in the Appendix. More complete integral tables are published separately. (Moderately complete tables are included in refs. 6 and 7 of Chap. 1. More extensive tabulations of integrals are those in refs. 9 and 10 of Chap. 1.)

3. Practice and experience in integrating various functions. In particular, it is important to be familiar with the organization and arrangement of the tables being used.

One difficulty concerning integration is that of knowing when to refer to the tables directly and when to try to convert a given integrand into some type form for which the integral is known. There are two extremes towards which the student might stray:

1. Trying to get along without tables. This should be considered only as an emergency measure when no tables are available, and even then it is of doubtful value. Intelligent use of tables of integrals should be an ability possessed by every student.

2. Excessive reliance upon tables. This results in waste of time in looking up integrals which should be written at sight.

Clearly, the best procedure is along a middle path between these two extremes.

There are many possible methods of attack which may be brought to bear upon an unknown integral form. For brevity, the treatment in this book is limited to simpler techniques, which, however, suffice for a great many functions. More extensive treatments of integration, as well as some proofs omitted here, are available elsewhere.[1,2]

15-2 Partial fractions. We consider here the problem of integrating *rational fractional* functions (ratios of polynomials). The basic idea, if the fraction is not already a type form, is to rewrite it as the *sum of simpler fractions*, each of which is a type form or readily reduced to a type form.

(If the given fraction is *improper*, that is, if the numerator contains powers of x at least as high as the highest power of x in the denominator, we first *divide* the numerator by the denominator according to the ordinary procedure of algebra. This reduces our problem to cases where the numerator is of lower degree.)

(*a*) *Denominator composed of linear factors not repeated.* Let it be desired to evaluate

$$\int \frac{(5x - 4)\,dx}{2x^2 + x - 6} \tag{1}$$

This is not readily integrated by methods which we have studied. We note, however, that the given denominator can be factored, giving

$$\int \frac{(5x - 4)\,dx}{2x^2 + x - 6} = \int \frac{(5x - 4)\,dx}{(2x - 3)(x + 2)} \tag{2}$$

Now, it may be shown that (by methods to be described later) the integrand in the right member of (2) is actually the sum of two simpler fractions:

$$\frac{5x - 4}{(2x - 3)(x + 2)} = \frac{1}{2x - 3} + \frac{2}{x + 2} \tag{3}$$

(You can confirm this readily by placing the terms in the right member over a common denominator.) The given integral may then be written

$$\begin{aligned}\int \frac{(5x - 4)\,dx}{2x^2 + x - 6} &= \int \frac{dx}{2x - 3} + 2\int \frac{dx}{x + 2} \\ &= \frac{1}{2}\ln(2x - 3) + 2\ln(x + 2) + C \\ &= \ln[(2x - 3)^{1/2}(x + 2)^2] + C\end{aligned}$$

Having *seen the value* of this method, we now learn *how to break up a fraction* like the left member of (3) into the sum of simpler *partial fractions* like the right member.* The denominators of the partial fractions are simply the factors of the original denominator. And it turns out that when these denominators contain x in only the first power, as in the above example, the numerators are *numbers*, not involving x. To illustrate the procedure, let us see *how* the left member of (3) was broken up into partial fractions given in the right member.

Assume that

$$\frac{5x - 4}{(2x - 3)(x + 2)} = \frac{A}{2x - 3} + \frac{B}{x + 2} \tag{4}$$

where A and B have values as yet unknown. Putting the right member of (4) over a common denominator,

$$\frac{5x - 4}{(2x - 3)(x + 2)} = \frac{A(x + 2) + B(2x - 3)}{(2x - 3)(x + 2)} = \frac{(A + 2B)x + 2A - 3B}{(2x - 3)(x + 2)}$$

Since the first and third members of this equation have similar denominators, their numerators must also be equal:

$$(A + 2B)x + 2A - 3B = 5x - 4$$

For this expression to be true the coefficients of x in both members must be equal, as must also the numerical terms. Thus

$$\begin{aligned} A + 2B &= 5 \\ 2A - 3B &= -4 \end{aligned}$$

Here we have two equations involving two unknowns, A and B. Solving them simultaneously,

$$A = 1 \qquad B = 2$$

Inserting these values in (4), we get the result (3).†

* Algebraic proof exists that rational fractions like that in (3) and of other kinds discussed in this section can be decomposed into partial fractions.[3] Here we *assume* that this operation is valid.

† A short cut which is useful in certain problems is described in ref. 1, p. 152, and in ref. 2, p. 253.

If the given denominator contains n different *linear* factors, that is, factors of the form $ax + b$, then the given fraction may be decomposed into n partial fractions whose denominators are the factors of the given denominator and whose numerators are numbers which can be determined.

Example 1. Evaluate

$$\int \frac{(11x^2 + 2x - 3)\,dx}{(x - 1)(x + 1)(3x + 2)}$$

The given integrand can be split into three partial fractions:

$$\frac{11x^2 + 2x - 3}{(x - 1)(x + 1)(3x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{3x + 2} \tag{5}$$

where A, B, and C are numbers to be found. Placing the right member over a common denominator and collecting like terms,

$$\frac{11x^2 + 2x - 3}{(x - 1)(x + 1)(3x + 2)} = \frac{(3A + 3B + C)x^2 + (5A - B)x + 2A - 2B - C}{(x - 1)(x + 1)(3x + 2)}$$

Equating coefficients of like powers of x in the numerators of these fractions,

$$\begin{aligned} 3A + 3B + C &= 11 \\ 5A - B \qquad &= 2 \\ 2A - 2B - C &= -3 \end{aligned}$$

Solving this system of simultaneous equations,*

$$A = 1 \qquad B = 3 \qquad C = -1$$

Substituting these values into (5),

$$\frac{11x^2 + 2x - 3}{(x - 1)(x + 1)(3x + 2)} = \frac{1}{x - 1} + \frac{3}{x + 1} - \frac{1}{3x + 2}$$

Thus the given integral is

$$\begin{aligned} \int \frac{(11x^2 + 2x - 3)\,dx}{(x - 1)(x + 1)(3x + 2)} &= \ln (x - 1) + 3 \ln (x + 1) - \frac{1}{3} \ln (3x + 2) + C \\ &= \ln \frac{(x - 1)(x + 1)^3}{(3x + 2)^{1/3}} + C \end{aligned}$$

(*b*) *Denominator containing quadratic factors not repeated.* Let it be desired to integrate

$$\int \frac{(x^2 - 13x + 9)\,dx}{(x + 2)(x^2 - 2x + 5)}$$

* In solving systems of three or more simultaneous equations, it is useful to know the method of *determinants*, described in texts on college algebra or analytic geometry.

The quadratic factor $x^2 - 2x + 5$ cannot be factored into *real* linear factors, so it is better to leave it in its present form. The integrand, it may be shown, is separable into partial fractions thus:

$$\frac{x^2 - 13x + 9}{(x + 2)(x^2 - 2x + 5)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 - 2x + 5} \tag{6}$$

Observe the manner of setting up these partial fractions:

1. Since the denominator $x + 2$ contains x in only the first power, we provide for it a numerator which is a simple number A, just as in part (a) of this section.

2. A numerator corresponding to a quadratic denominator like $x^2 - 2x + 5$ must have two terms, one of which contains x, the other of which is a number.

Putting the right member of (6) over a common denominator and collecting like terms,

$$\frac{x^2 - 13x + 9}{(x + 2)(x^2 - 2x + 5)} = \frac{(A + B)x^2 - (2A - 2B - C)x + 5A + 2C}{(x + 2)(x^2 - 2x + 5)}$$

Equating like powers of x in the denominators of these members,

$$\begin{aligned} A + B &= 1 \\ -2A + 2B + C &= -13 \\ 5A + 2C &= 9 \end{aligned}$$

or

$$A = 3 \qquad B = -2 \qquad C = -3$$

Substituting these values in (6), we get for the given integral

$$\int \frac{(x^2 - 13x + 9)\,dx}{(x + 2)(x^2 - 2x + 5)} = \int \frac{3dx}{x + 2} - \int \frac{(2x + 3)\,dx}{x^2 - 2x + 5}$$

The first of these resulting integrals gives $3 \ln (x + 2) + C_1$. The second can be written

$$\begin{aligned} \int \frac{(2x + 3)\,dx}{x^2 - 2x + 5} &= \int \frac{(2x - 2)\,dx}{x^2 - 2x + 5} + 5 \int \frac{dx}{x^2 - 2x + 5} \\ &= \ln (x^2 - 2x + 5) + \frac{5}{2} \tan^{-1} \frac{x - 1}{2} + C_2 \end{aligned}$$

so that

$$\int \frac{(x^2 - 13x + 9)\,dx}{(x + 2)(x^2 - 2x + 5)}$$
$$= 3 \ln (x + 2) - \ln (x^2 - 2x + 5) - \frac{5}{2} \tan^{-1} \frac{x - 1}{2} + C$$

where $C = C_1 - C_2$. This result can be written

$$\ln \frac{(x+2)^3}{x^2 - 2x + 5} - \frac{5}{2} \tan^{-1} \frac{x-1}{2} + C$$

Where there are several different quadratic denominators, there will, in general, be a different linear numerator (of the form $ax + b$) for each such denominator. For example, the fraction in the left member below would be broken up as shown:

$$\frac{3x^4 + 6x^3 + 17x^2 + 28x - 20}{(x+3)(x^2 - 4x + 5)(x^2 + 2x + 2)} = \frac{A}{x+3} + \frac{Bx + C}{x^2 - 4x + 5} + \frac{Dx + E}{x^2 + 2x + 2}$$

(c) *Repeated factors.* Suppose a factor appears m times in the denominator of a given fraction. Here we provide appropriate partial fractions having as their denominators the given factor in all whole-number powers up to and including the mth.

Example 2. Separate into partial fractions

$$\frac{x^2 + 7x}{(x-1)(x+1)^2}$$

According to the rule just stated,

$$\frac{x^2 + 7x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Placing the right member over a common denominator and collecting like terms,

$$\frac{x^2 + 7x}{(x-1)(x+1)^2} = \frac{(A+B)x^2 + (2A + C)x + A - B - C}{(x-1)(x+1)^2}$$

Equating like powers of x in the two numerators,

$$\begin{aligned} A + B \qquad &= 1 \\ 2A \qquad + C &= 7 \\ A - B - C &= 0 \end{aligned}$$

or $\qquad A = 2 \qquad B = -1 \qquad C = 3$

Therefore
$$\frac{x^2 + 7x}{(x-1)(x+1)^2} = \frac{2}{x-1} - \frac{1}{x+1} + \frac{3}{(x+1)^2}$$

Example 3. Separate into partial fractions

$$\frac{x^2 - 2x - 2}{(x^2 - x + 1)^2}$$

This may be separated as

$$\frac{x^2 - 2x - 2}{(x^2 - x + 1)^2} = \frac{Ax + B}{x^2 - x + 1} + \frac{Cx + D}{(x^2 - x + 1)^2}$$

This gives

$$\begin{aligned} A \qquad\qquad &= 0 \\ -A + B \qquad &= 1 \\ A - B + C \quad &= -2 \\ B \qquad + D &= -2 \end{aligned}$$

so $\qquad A = 0 \qquad B = 1 \qquad C = -1 \qquad D = -3$

Therefore
$$\frac{x^2 - 2x - 2}{(x^2 - x + 1)^2} = \frac{1}{x^2 - x + 1} - \frac{x + 3}{(x^2 - x + 1)^2}$$

PROBLEMS

In Probs. 1 to 13 integrate after separating the integrands into partial fractions.

1. $\displaystyle\int \frac{dx}{x^2 + 2x - 3}$
2. $\displaystyle\int \frac{x\,dx}{x^2 - 1}$
3. $\displaystyle\int \frac{dx}{x(x + 1)}$
4. $\displaystyle\int \frac{5dx}{x^3 + 5x}$
5. $\displaystyle\int \frac{(5x + 1)\,dx}{x^2 + x - 2}$
6. $\displaystyle\int \frac{dx}{x^2 - 7x + 12}$
7. $\displaystyle\int \frac{(x - 3)\,dx}{6x^2 - x - 1}$
8. $\displaystyle\int \frac{(3x^2 - 2)\,dx}{(x + 1)(x^2 + x + 1)}$
9. $\displaystyle\int \frac{(2x + 1)\,dx}{(3x - 1)(x^2 + 2x + 2)}$
10. $\displaystyle\int \frac{(x + 1)\,dx}{(x - 1)(x^2 + 1)}$
11. $\displaystyle\int \frac{(2x + 1)\,dx}{(x - 1)(x - 3)^2}$
12. $\displaystyle\int \frac{(2x + 5)\,dx}{(x + 3)(x + 1)^2}$
13. $\displaystyle\int \frac{(x^2 + x + 2)\,dx}{(x - 1)^3}$

14. Decompose into partial fractions

$$\frac{3x^4 + 6x^3 + 17x^2 + 28x - 20}{(x + 3)(x^2 - 4x + 5)(x^2 + 2x + 2)}$$

[HINT: The start of the work is shown at the conclusion of part (*b*) of the preceding section of the text.]

15. Over a certain interval the induced emf in a 3-henry inductor varied as $v_{ind} = (5t - 4)/(t^2 - t - 2)$ volts. What must have been the formula for the current in the inductor?

16. A solenoid moved a rod with a speed $v = (t + 3)/(t^2 + 3t + 2)$ centimeters per second. Over what distance did the rod move from $t = 0$ to $t = 2$ seconds?

17. The current in a circuit varied, over a certain interval, according to $i = (t^2 - t + 1)/(t + 1)(t + 2)^2$ amperes. Find a formula for the charge transmitted from $t = 0$ to $t = t_1$ seconds.

18. A mutual inductance $M = 3$ henrys exists between two windings. Find current i_1 which must be forced through one of these windings to produce in the other an induced emf $v_2 = (3t^2 + 13t + 13)/(t + 2)^2(t + 3)$ volts.

15-3 Further trigonometric and hyperbolic forms.

Many trigonometric integrands are readily reduced to type forms through the use of identities (Table 1). Several cases are illustrated below.

a. $\int \sin^2 u\, du$ *and* $\int \cos^2 u\, du$. These forms integrate readily if we substitute $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$ or $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$.

Example 1. A sine wave of current $i = I_{\max} \sin \omega t$ flows in a circuit. Show that the effective or rms value of the current is

$$I_{\text{eff}} = \tfrac{1}{2}\sqrt{2}\, I_{\max} \tag{7}$$

The effective value of an alternating current is that value of direct current which produces an equal heating effect. When the current of (7) flows in a resistance R, heat is produced according to the power formula $p = i^2R$. The average power in R, over an interval of 1 cycle, will then be equal to the area under the graph of the function i^2R divided by the length of the interval:

$$I_{\text{eff}}^2 R = \frac{1}{2\pi}\int_0^{2\pi} i^2R\, d(\omega t) = \frac{I_{\max}^2 R}{2\pi}\int_0^{2\pi} \sin^2 \omega t\, d(\omega t)$$

or

$$I_{\text{eff}} = \sqrt{\frac{I_{\max}^2}{2\pi}\int_0^{2\pi} \sin^2 \omega t\, d(\omega t)}$$

Substituting $\sin^2 \omega t = \frac{1}{2}(1 - \cos 2\omega t)$,

$$I_{\text{eff}} = \sqrt{\frac{I_{\max}^2}{2\pi}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2\omega t\right) d(\omega t)} = \sqrt{\frac{I_{\max}^2}{2\pi}\left(\frac{1}{2}\omega t - \frac{1}{4}\sin 2\omega t\right)\Big]_{\omega t=0}^{2\pi}}$$

$$= \frac{I_{\max}}{\sqrt{2}} = \frac{1}{2}\sqrt{2}\, I_{\max}$$

b. $\int \tan^2 u\, du$ *and* $\int \cot^2 u\, du$. Here we use, respectively, the identities $\tan^2 u = \sec^2 u - 1$, or $\cot^2 u = \csc^2 u - 1$.

Example 2. Evaluate $\int \tan^2 \theta\, d\theta$.

Substituting $\tan^2 \theta = \sec^2 \theta - 1$,

$$\int \tan^2 \theta\, d\theta = \int \sec^2 \theta\, d\theta - \int d\theta$$

From Table 11, we find that $\int \sec^2 \theta\, d\theta = \tan \theta$ (+ constant). Then

$$\int \tan^2 \theta\, d\theta = \tan \theta - \theta + C.$$

c. $\int \sin^m u \cos^n u\, du$ *and* $\int \cos^m u \sin^n u\, du$ (*m odd and positive*). Factor out $-\sin u$ or $\cos u$, respectively. Then use $\sin^2 u + \cos^2 u = 1$.

Example 3. Find $\int \sin^5 x \cos^2 x\, dx$.
Write

$$\int \sin^5 x \cos^2 x\, dx = -\int \sin^4 x \cos^2 x(-\sin x\, dx)$$

Since $\sin^2 u + \cos^2 u = 1$, $\sin^4 x$ can be written $(1 - \cos^2 x)^2$:

$$\begin{aligned} \int \sin^5 x \cos^2 x\, dx &= -\int (1 - \cos^2 x)^2 \cos^2 x(-\sin x\, dx) \\ &= -\int (\cos^2 x - 2\cos^4 x + \cos^6 x)(-\sin x\, dx) \end{aligned}$$

Noting that $-\sin x\, dx$ is the differential of $\cos x$,

$$\int \sin^5 x \cos^2 x\, dx = -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C$$

d. $\int \tan^m u \sec^n u\, du$ *and* $\int \cot^m u \csc^n u\, du$ (*m odd and positive*). Factor out $\tan u \sec u$ or $-\cot u \csc u$, respectively. Then use $\tan^2 u = \sec^2 u - 1$ or $\cot^2 u = \csc^2 u - 1$.

Example 4. Evaluate $\int \cot^7 \phi \csc^4 \phi\, d\phi$.
Factoring out $-\cot \phi \csc \phi$,

$$\int \cot^7 \phi \csc^4 \phi\, d\phi = -\int \cot^6 \phi \csc^3 \phi\, (-\cot \phi \csc \phi\, d\phi)$$

Substituting $\cot^2 \phi = \csc^2 \phi - 1$,

$$\begin{aligned} \int \cot^7 \phi \csc^4 \phi\, d\phi &= -\int (\csc^2 \phi - 1)^3 \csc^3 \phi(-\cot \phi \csc \phi\, d\phi) \\ &= -\int (\csc^9 \phi - 3\csc^7 \phi + 3\csc^5 \phi - \csc^3 \phi)(-\cot \phi \csc \phi\, d\phi) \end{aligned}$$

Since $-\csc \phi \cot \phi\, d\phi = d(\csc \phi)$, this gives

$$\int \cot^7 \phi \csc^4 \phi\, d\phi = -\tfrac{1}{10}\csc^{10} \phi + \tfrac{3}{8}\csc^8 \phi - \tfrac{1}{2}\csc^6 \phi + \tfrac{1}{4}\csc^4 \phi + C$$

e. $\int \sec^m u \tan^n u\, du$ and $\int \csc^m u \cot^n u\, du$ (*m even and positive*). Factor out $\sec^2 u$ or $-\csc^2 u$, respectively; then use

$$\sec^2 u = 1 + \tan^2 u \qquad \text{or} \qquad \csc^2 u = 1 + \cot^2 u$$

Example 5. Evaluate $\int \sec^4 u \tan^6 u\, du$.
Let this be written

$$\begin{aligned} \int \sec^4 u \tan^6 u\, du &= \int \sec^2 u \tan^6 u\, (\sec^2 u\, du) = \int (1 + \tan^2 u) \tan^6 u(\sec^2 u\, du) \\ &= \int (\tan^6 u + \tan^8 u)\, (\sec^2 u\, du) \end{aligned}$$

Since $\sec^2 u\, du = d(\tan u)$,

$$\int \sec^4 u \tan^6 u\, du = \tfrac{1}{7}\tan^7 u + \tfrac{1}{9}\tan^9 u + C$$

Note that, in parts (*c*), (*d*), and (*e*) of this section, the procedures apply even if $n = 0$, that is, even if the second factor in the integrand is absent.

f. Integrals of hyperbolic expressions. Procedures similar to those just outlined permit us to integrate many expressions involving hyperbolic functions, through use of hyperbolic identities (Table 7).

Example 6. Evaluate $\int \sinh^3 u \cosh^4 u\, du$.
Factor out $\sinh u$:

$$\begin{aligned}\int \sinh^3 u \cosh^4 u\, du &= \int \sinh^2 u \cosh^4 u\, (\sinh u\, du)\\ &= \int (\cosh^2 u - 1) \cosh^4 u\, (\sinh u\, du)\\ &= \int (\cosh^6 u - \cosh^4 u)(\sinh u\, du)\\ &= \tfrac{1}{7} \cosh^7 u - \tfrac{1}{5} \cosh^5 u + C\end{aligned}$$

PROBLEMS

In Probs. 1 to 18 evaluate the given integrals.

1. $\int \cos^2 \theta\, d\theta$
2. $\int \cot^2 x\, dx$
3. $\int \sin^3 x \cos^2 x\, dx$
4. $\int \sin^5 x \cos^4 x\, dx$
5. $\int \cos^3 \phi \sin^4 \phi\, d\phi$
6. $\int \tan^3 x \sec^2 x\, dx$
7. $\int \cot^5 z \csc^4 z\, dz$
8. $\int \sec^2 \theta \tan^3 \theta\, d\theta$
9. $\int \sec^6 \theta \tan^5 \theta\, d\theta$
10. $\int \csc^4 x \cot^3 x\, dx$
11. $\int x \sin^2 x^2\, dx$
12. $\int \sinh^2 x\, dx$
13. $\int \cosh^2 x\, dx$
14. $\int \tanh^2 \alpha\, d\alpha$
15. $\int \sinh^3 x \cosh^2 x\, dx$
16. $\int \cosh^5 x \sinh^2 x\, dx$
17. $\int \tanh^3 x\, \mathrm{sech}^4 x\, dx$
18. $\int \mathrm{csch}^2 x \coth^4 x\, dx$

19. Evaluate $\int \sin^3 x \cos^5 x\, dx$ in two ways: (a) by first factoring out $-\sin x$ and (b) by first factoring out $\cos x$. Account for the different appearances of the results (Sec. 9-9).

20. The *cardioid* radiation pattern of a certain antenna is given approximately by $\mathbf{E} = k(1 - \cos \theta)$, where $\mathbf{E}$ is the field intensity at a given distance, k is a constant, and θ is the angle indicating the direction from the antenna in which the intensity is measured. Find the rms value of the pattern, that is, the radius of a circle having an area equal to the area of the polar graph of the pattern.

21. In studying rf amplifiers it was desired to find the *crest factor* (ratio of peak to average value) of the wave $i = \sin^2 \theta$ over a $\frac{1}{2}$-cycle interval. Determine this crest factor.

22. A current $i = 2 \tan \omega t$ flows in a 5-ohm resistor. Find the energy dissipated in the resistor from $\omega t = 0$ to $\omega t = \pi/4$, where i is in amperes and t is in seconds.

23. Find the total charge transmitted during the interval from $\omega t = 0$ to $\omega t = \pi/2$ when the current is $i = \cos^3 \omega t \sin^5 \omega t$ amperes.

15-4 Integration by parts.

15-4 Integration by parts. A very useful integration method, called *integration by parts*, is obtained from the formula for the derivative of a product:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Multiplying each term by dx and rearranging,

$$u\, dv = d(uv) - v\, du$$

Integrating each term,

$$\int u\, dv = uv - \int v\, du \tag{8}$$

In this result, which is a formula for integration by parts, the integral of the product $\int u\, dv$ (here, u and v are assumed to be functions of the same variable x) can be found if we can evaluate $\int v\, du$. And often the

latter integral can be made easier to evaluate than $\int u\,dv$ is. As an example, let us find

$$\int x \sin x\,dx \tag{9}$$

Here we may let

$$u = x \qquad dv = \sin x\,dx \tag{10}$$

Differentiating the first equation in (10) and integrating the second,

$$du = dx \qquad v = -\cos x \tag{11}$$

(The constant of integration for the latter expression will be provided shortly.) Substituting (10) and (11) in (8),

$$\int x \sin x\,dx = -x \cos x + \int \cos x\,dx \tag{12}$$

The last term of (12) integrates easily, giving

$$\int x \sin x\,dx = \sin x - x \cos x + C$$

where C includes the constants of integration for both (11) and (12).

Example 1. Find a formula for $\int \ln x\,dx$.
Let $u = \ln x$, $dv = dx$. Then $du = dx/x$, $v = x$. Putting these values in (8), we have $\int \ln x\,dx = x \ln x - \int dx$, or

$$\int \ln x\,dx = x \ln x - x + C \tag{13}$$

Although this method does not succeed in the case of every product that might be written, it is very powerful. Success in its use depends upon a *favorable choice of parts* (which portion to call u and which to call dv) in the given integrand. Clearly dv must be chosen such that it can be integrated. And u, if possible, should be such that $\int v\,du$ is simpler than the given integral $\int u\,dv$. (There are no rules for choice of parts, but a poor choice quickly becomes evident. A good start is often made simply by choosing u such that du is simpler than u.)

Example 2. Find $y = \int x^2 \cosh x\,dx$.
Several choices of parts are possible:

	$u = x^2 \cosh x$	$dv = dx$
or	$u = x \cosh x$	$dv = x\,dx$
or	$u = \cosh x$	$dv = x^2\,dx$
or	$u = x$	$dv = x \cosh x\,dx$
or	$u = x^2$	$dv = \cosh x\,dx$

The final choice makes it possible to find $\int dv$, and at the same time it gives $\int v\,du$ simpler than the given integral. Letting, then, $u = x^2$, $dv = \cosh x\,dx$, we have $du = 2x\,dx$, $v = \sinh x$. This makes

$$y = \int x^2 \cosh x\,dx = x^2 \sinh x - 2\int x \sinh x\,dx$$

To evaluate the integral in the right member, we *again* use integration by parts, taking $u = x$, $dv = \sinh x\, dx$, so that $du = dx$, $v = \cosh x$. This makes

$$\int x \sinh x\, dx = x \cosh x - \int \cosh x\, dx = x \cosh x - \sinh x + C$$

so that

$$y = x^2 \sinh x - 2x \cosh x + 2 \sinh x + C_1 = (x^2 + 2) \sinh x - 2x \cosh x + C_1$$

When $\int v\, du$ cannot be integrated at once, the original integral can sometimes be found by *solving an equation.* An example follows.

Example 3. In studying a current wave resulting when an ac voltage is suddenly applied to a series RL circuit it was necessary to evaluate $\int e^{ax} \sin bx\, dx$. Carry out this integration.

Let us try $u = e^{ax}$, $dv = \sin bx\, dx$, so that $du = ae^{ax}\, dx$, $v = -(1/b) \cos bx$. Then

$$\int e^{ax} \sin bx\, dx = -\frac{e^{ax}}{b} \cos bx + \frac{a}{b} \int e^{ax} \cos bx\, dx$$

Applying integration by parts to the integral on the right, we let $u = e^{ax}$, $dv = \cos bx\, dx$, so that $du = ae^{ax}\, dx$, $v = (1/b) \sin bx$:

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{b} \sin bx - \frac{a}{b} \int e^{ax} \sin bx\, dx$$

so $$\int e^{ax} \sin bx\, dx = -\frac{e^{ax}}{b} \cos bx + \frac{ae^{ax}}{b^2} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx\, dx$$

The last term is the same as the given integral, multiplied by $-a^2/b^2$. Adding $(a^2/b^2)\int e^{ax} \sin bx\, dx$ to each side of the equation,

$$\left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \sin bx\, dx = \frac{ae^{ax}}{b^2} \sin bx - \frac{e^{ax}}{b} \cos bx + C$$

Dividing by $1 + a^2/b^2$ and simplifying,

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C_1$$

PROBLEMS

Evaluate the integrals in these problems, using integration by parts.

1. $\int x \cos x\, dx$
2. $\int x \sinh x\, dx$
3. $\int x \ln x\, dx$
4. $\int x e^x\, dx$
5. $\int x^2 \sin x\, dx$
6. $\int x^2 \sinh x\, dx$
7. $\int x^2 e^x\, dx$
8. $\int x^2 \ln x\, dx$
9. $\int e^x \cos x\, dx$
10. $\int x^2 \sin 2x\, dx$
11. $\int \sin^{-1} x\, dx$
12. $\int \cos^{-1} x\, dx$
13. $\int \tanh^{-1} x\, dx$

14. Evaluate the integral of Example 3, this time letting $u = \sin bx$ and $dv = e^{ax}\, dx$.

15. During a certain interval the current in a circuit varied according to $i = 9t \sin 3t$ amperes. Find a formula for the charge transferred during an interval from $t = 0$ to $t = t_1$ seconds.

16. What current wave must be sent through the primary of a transformer to produce an induced emf $v_2 = t^2e^{2t}$ volts in the secondary if the mutual inductance between windings is $\frac{1}{2}$ henry?

17. Same as Prob. 16, except let $v_2 = t \cosh 2t$ volts.

18. What energy is delivered to a 10-ohm resistor during the time from $t = 0$ to $t = \pi/4$ second if the current is $i = 2t \sin t$ amperes?

19. Over a certain interval the current supplied to a 10-microfarad capacitor was $i_C = 2t^2 \ln 50t$ amperes. Neglecting resistance, what must have been the applied-voltage formula?

15-5 Substitutions. Temporarily substituting a new variable can often reduce a given integral to a type form. Sometimes the details of the substitution can be carried out mentally. We should make sure, before performing a substitution, that the substitution is *necessary;* that is, we should not waste time in making substitutions when the given integral is already a type form. While there are no rules to cover all cases, the following cases should be studied.

a. Integrands involving linear radicals. Tables alone are not usually sufficient for finding the integrals of expressions involving radicals if there are also present functions of x outside the radical. In all cases where the quantity inside the radical is of the form $ax + b$, and in some other cases, the expression can be reduced to a rational form by substituting a new variable z for the radical.

Example 1. Find

$$\int \frac{\sqrt{x+5}\,dx}{x+6}$$

Substituting $z = (x + 5)^{\frac{1}{2}}$, we get $x = z^2 - 5$, $dx = 2z\,dz$, so that

$$\int \frac{\sqrt{x+5}\,dx}{x+6} = 2\int \frac{z^2\,dz}{z^2+1}$$

This may be written

$$\int \frac{\sqrt{x+5}\,dx}{x+6} = 2\int \frac{(z^2+1-1)\,dz}{z^2+1} = 2\int dz - 2\int \frac{dz}{z^2+1}$$
$$= 2z - 2\tan^{-1} z + C$$

Reversing the original substitution, we put $(x + 5)^{\frac{1}{2}}$ in place of z:

$$\int \frac{\sqrt{x+5}\,dx}{x+6} = 2\sqrt{x+5} - 2\tan^{-1}\sqrt{x+5} + C$$

b. Integrands containing powers of $ax + b$. Try setting $ax + b$ equal to a suitable power of z.

Example 2. Evaluate

$$\int \frac{x\,dx}{(2x+5)^{3/2}}$$

Let $z^2 = 2x + 5$. Then $z = (2x + 5)^{1/2}$, and $z^3 = (2x + 5)^{3/2}$. Also $2z\,dz = 2dx$, or $dx = z\,dz$. And $x = \frac{1}{2}(z^2 - 5)$. Making these substitutions,

$$\int \frac{x\,dx}{(2x+5)^{3/2}} = \frac{1}{2}\int \frac{(z^2-5)z\,dz}{z^3} = \frac{1}{2}\int dz - \frac{5}{2}\int z^{-2}\,dz = \frac{1}{2}z + \frac{5}{2}z^{-1} + C$$

Reversing the substitution,

$$\int \frac{x\,dx}{(2x+5)^{3/2}} = \frac{1}{2}(2x+5)^{1/2} + \frac{5}{2}(2x+5)^{-1/2} + C = \frac{x+5}{(2x+5)^{1/2}} + C$$

c. *Integrands containing powers of* $ax^n + b$. Try setting $ax^n + b$ equal to some suitable power of z.

Example 3. Find

$$\int \frac{x^3\,dx}{(x^2-4)^2}$$

Let $z = x^2 - 4$. From this,

$$dz = 2x\,dx \qquad (14)$$

and

$$x^2 = z + 4 \qquad (15)$$

But we need an expression for $x^3\,dx$ in terms of z. Multiplying (14) by (15) and dividing the result by 2,

$$x^3\,dx = \tfrac{1}{2}(z+4)\,dz$$

Then

$$\int \frac{x^3\,dx}{(x^2-4)^2} = \frac{1}{2}\int \frac{(z+4)\,dz}{z^2} = \frac{1}{2}\int \frac{dz}{z} + 2\int z^{-2}\,dz$$

$$= \ln z^{1/2} - \frac{2}{z} + C = \ln (x^2-4)^{1/2} - \frac{2}{x^2-4} + C$$

Example 4. Find

$$\int \frac{(2x^2-9)^{3/2}\,dx}{x}$$

Let $z^2 = 2x^2 - 9$. This gives

$$z = (2x^2-9)^{1/2} \qquad (16)$$

$$z^3 = (2x^2-9)^{3/2} \qquad (17)$$

$$\tfrac{1}{2}(z^2+9) = x^2 \qquad (18)$$

$$2z\,dz = 4x\,dx \qquad (19)$$

We want an expression for dx/x. This is obtained by dividing (19) by 4 and by (18):

$$\frac{z\,dz}{z^2+9} = \frac{dx}{x}$$

Thus

$$\int \frac{(2x^2-9)^{3/2}\,dx}{x} = \int \frac{z^3 z\,dz}{z^2+9} = \int \frac{z^4\,dz}{z^2+9} = \int \left(z^2 - 9 + \frac{81}{z^2+9}\right) dz$$

$$= \frac{1}{3} z^3 - 9z + 27 \tan^{-1} \frac{z}{3} + C$$

$$= \frac{1}{3} (2x^2-9)^{3/2} - 9(2x^2-9)^{1/2} + 27 \tan^{-1} \frac{(2x^2-9)^{1/2}}{3} + C$$

$$= \frac{1}{3} (2x^2-9)^{1/2}(2x^2-36) + 27 \tan^{-1} \frac{(2x^2-9)^{1/2}}{3} + C$$

PROBLEMS

Solve by finding suitable substitutions.

1. $\int (x+2)\sqrt{x-3}\,dx$
2. $\int (x^2+2x)\sqrt{x-1}\,dx$
3. $\displaystyle\int \frac{dx}{(x+2)\sqrt{x-2}}$
4. $\displaystyle\int \frac{(x^2+4x)\,dx}{\sqrt{x+6}}$
5. $\displaystyle\int \frac{x\,dx}{(2x+1)^{1/3}}$
6. $\int x(1+x)^{2/3}\,dx$
7. $\displaystyle\int \frac{x^2\,dx}{(x+1)^{1/3}}$
8. $\displaystyle\int \frac{x^2\,dx}{(3x+2)^{2/3}}$
9. $\displaystyle\int \frac{x\,dx}{(2x+3)^{3/4}}$
10. $\displaystyle\int \frac{x^3\,dx}{(x^2+16)^2}$
11. $\int x^5(x^2-4)^3\,dx$
12. $\int x^5(x^3+2)^{1/2}\,dx$
13. $\displaystyle\int \frac{dx}{x(x^2-4)^{1/2}}$
14. $\displaystyle\int \frac{(x^2-9)^{5/2}\,dx}{x}$

15. The current in a circuit varied according to $i = t(t+2)^{1/2}$ amperes. Find the charge transmitted during the interval from $t = 0$ to $t = 1$ second.

16. If the mutual inductance between the windings of a transformer is M henrys, what primary current i_1 must flow so that the induced secondary emf will be $v_2 = -(1/M)t^2(t+1)^{3/2}$ volts?

17. Same as Prob. 16, except that the desired secondary voltage will be $v_2 = -(1/M)\,[t/(t^2+36)^{1/2}]$ volts.

18. The power in a circuit varied according to $p = t^3/(t^2+9)^2$ watts. How much energy was consumed from $t = 0$ to $t = 1$ second?

19. The voltage applied to a circuit was $v = 1/t$ volts. The circuit constants varied so that the current was $i = (t^2-4)^{3/2}$ amperes. What energy was used from $t = 2$ to $t = T$ seconds?

15-6 Improper integrals. *a. Infinite intervals.* In many electrical problems we need the idea of a definite integral in which one or both

of the limits is not finite. Consider

$$\int_1^\infty \frac{dx}{x^3} \tag{20}$$

Graphically, we might try to picture this integral as the area (Fig. 15-1) beneath the graph $y = x^{-3}$ over the interval from $x = 1$ to $x = \infty$. But such a notion, of itself, is meaningless. For no matter how far we might go along the x axis, a still greater area could always be obtained by letting x increase even further. Thus a new definition is needed if we are to attach meaning to integrals such as (20) in which the limits are infinite. It is entirely possible for the area beneath the graph of the integrand to *approach a limit* as x increases without bound, and it is this limit, if it exists, which we take as the desired integral. That is:

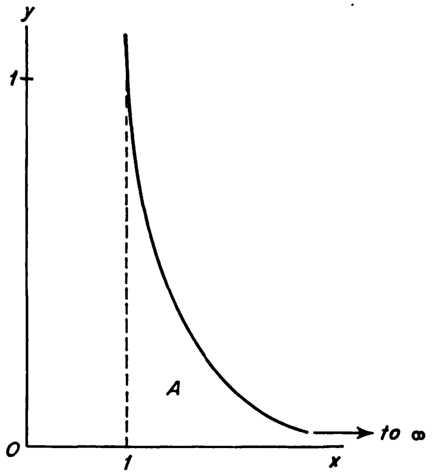


Fig. 15-1

➠ We define the integral of $f(x)\,dx$ from a to ∞ as the limit approached by the integral of $f(x)\,dx$ from $x = a$ to $x = q$, as $q \to \infty$ if this limit exists.

Thus, in the case of (20),

$$\int_1^\infty \frac{dx}{x^3} = \lim_{q\to\infty} \int_1^q \frac{dx}{x^3} = \lim_{q\to\infty} \left(-\frac{1}{2}x^{-2}\right)\Big]_1^q = \lim_{q\to\infty}\left(-\frac{1}{2}q^{-2} + \frac{1}{2}\right) = \frac{1}{2}$$

A corresponding definition applies if the lower limit is not finite. For example,

$$\int_{-\infty}^0 e^x\,dx = \lim_{p\to-\infty} \int_p^0 e^x\,dx = \lim_{p\to-\infty} e^x\Big]_p^0 = \lim_{p\to-\infty} (e^0 - e^p) = 1$$

A definite integral in which one (or both) of the limits is not finite is called an *improper integral.* If it has a value as defined above, that is, if the area under the graph of the integrand approaches a limit, we say that the integral *converges.* If, however, this limit does not exist, we say that the improper integral *diverges* or that *it does not exist,* and for our purposes we say that it has no value.

Example 1. Does $\int_1^\infty dx/x$ exist? If so, what is its value?

We write

$$\int_1^\infty \frac{dx}{x} = \lim_{q\to\infty} \int_1^q \frac{dx}{x} = \lim_{q\to\infty} \ln x \Big]_1^q = \lim_{q\to\infty} (\ln q - \ln 1) = \infty$$

Therefore the integral is divergent.

Example 2. A capacitor of capacitance C is charged to a voltage V. Show that the energy then contained in the capacitor is $W = CV^2/2$ joules.

To accomplish this, suppose the capacitor to contain zero charge at time $t = 0$, and suppose that at that instant it is connected to a source having a voltage V. The charging current through the resistance R which inevitably exists in the circuit will be

$$i = \frac{V}{R} e^{-t/RC} \qquad \text{amperes}$$

By Kirchhoff's voltage law, the sum of the voltage drops (v_C across the capacitor, v_R across the resistance in the circuit, and the negative "drop" of the source voltage V) must equal zero:

$$v_C + v_R - V = 0 \qquad \text{or} \qquad v_C = V - v_R$$

Since $v_R = Ri$, we have

$$v_C = V - Ri \qquad \text{volts}$$

The power being expended at any instant in charging the capacitor is then

$$p_C = v_C i = Vi - Ri^2 = \frac{V^2}{R} e^{-t/RC} - \frac{V^2}{R} e^{-2t/RC} \qquad \text{watts}$$

Since $W = \int_0^t p_C\, dt$ is the energy used, up to any time t, in charging the capacitor, the limiting value of energy approached as time goes on (and as the capacitor voltage approaches V) will be

$$\begin{aligned} W &= \frac{V^2}{R} \int_0^\infty e^{-t/RC}\, dt - \frac{V^2}{R} \int_0^\infty e^{-2t/RC}\, dt \\ &= \frac{V^2}{R} \lim_{q\to\infty} \left(\int_0^q e^{-t/RC}\, dt - \int_0^q e^{-2t/RC}\, dt \right) \\ &= -CV^2 \left[(0 - 1) - \left(0 - \frac{1}{2}\right) \right] = \frac{CV^2}{2} \qquad \text{joules} \end{aligned}$$

Example 3. The *electric potential* V volts at any point P is defined as the amount of work required to bring a positive charge of one coulomb from an indefinitely great distance up to the point P. Derive a formula for the potential at P due to a positive charge of Q coulombs located at distance S from P.

According to Coulomb's law, the force $\mathbf{F}$ newtons acting between two like

charges Q_1 and Q_2 coulombs is

$$\mathbf{F} = -\frac{Q_1Q_2}{4\pi\epsilon s^2} \qquad \text{newtons}$$

where s is the distance separating them and ϵ is the permittivity of the medium in which they are situated. Since the work $W = \int \mathbf{F}\, ds$, the potential at P is, by the above definition,

$$V = -\int_{s=\infty}^{S} \frac{Q}{4\pi\epsilon s^2}\, ds = -\frac{Q}{4\pi\epsilon} \lim_{p\to\infty} \left(\int_{s=p}^{S} s^{-2}\, ds\right) = \lim_{p\to\infty} \frac{Q}{4\pi\epsilon s}\bigg]_{s=p}^{S}$$
$$= \frac{Q}{4\pi\epsilon S} \qquad \text{volts}$$

b. Integrand infinite. A further type which also bears the name *improper integral* is that in which the integrand becomes infinite in the interval of integration.

CASE I. Suppose the *integrand becomes infinite at the upper limit.* For example, consider

$$\int_0^1 \frac{dx}{\sqrt{1 - x^2}} \tag{21}$$

Here, the integrand becomes infinite when $x = 1$, so that no area is defined by the graph of the integral. However, we consider the possibility that this area might approach a limit as $x \to 1$, and if it does, we take this limit to represent the value of the integral.

➢ We define the integral of $f(x)\, dx$ from a to b, if $f(x)$ becomes infinite when $x = b$, as the limit approached by the integral of $f(x)\, dx$ from a to q as $q \to b$, if this limit exists.

(For this calculation, we let q approach b from the *negative* side; that is, q increases toward the value of b.)

Evaluating (21) according to this definition,

$$\int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \lim_{q\to 1} \int_0^q \frac{dx}{\sqrt{1 - x^2}} = \lim_{q\to 1} \sin^{-1} x\bigg]_0^q = \frac{\pi}{2}$$

CASE II. Next we consider the case where the *integrand becomes infinite at the lower limit.*

➢ We define the integral of $f(x)\, dx$ from a to b, if $f(x)$ becomes infinite when $x = a$, as the limit approached by the integral of $f(x)\, dx$ from p to a, as $p \to a$, if this limit exists.

(Here, we let p approach a from the *positive* side; that is, p decreases toward the value of a.)

Example 4. Evaluate $\int_0^1 dx/\sqrt{x}$.

The integrand becomes infinite when $x = 0$, so we define

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{p \to 0} \int_p^1 \frac{dx}{\sqrt{x}} = \lim_{p \to 0} 2\sqrt{x}\Big]_{x=p}^1 = 2$$

Example 5. Evaluate

$$\int_{-1}^1 \frac{du}{\sqrt{1-u^2}}$$

Here the integrand is infinite at *both* the upper and the lower limits. We apply the definitions of both Case I and Case II:

$$\int_{-1}^1 \frac{du}{\sqrt{1-u^2}} = \lim_{\substack{p \to -1 \\ q \to 1}} \int_p^q \frac{du}{\sqrt{1-u^2}}$$

if the latter limit exists. (Here we let q approach the upper limit 1 from the *negative* side and q approach the lower limit -1 from the *positive* side.) This gives

$$\int_{-1}^1 \frac{du}{\sqrt{1-u^2}} = \lim_{\substack{p \to -1 \\ q \to 1}} \sin^{-1} u\Big]_{u=p}^q = \lim_{\substack{p \to -1 \\ q \to 1}} (\sin^{-1} q - \sin^{-1} p)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

CASE III. The integrand may *become infinite for some value of x between the upper and the lower limits.*

Suppose that, in the integral of $f(x)\,dx$ from a to b, the integrand $f(x)$ becomes infinite at some point $x = c$ lying between a and b. Then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

if both the latter integrals exist.

Example 6. Evaluate $\int_{-1}^1 x^{-4/5}\,dx$ if it exists.

The integrand becomes infinite when $x = 0$. Therefore we define

$$\int_{-1}^1 x^{-4/5}\,dx = \int_{-1}^0 x^{-4/5}\,dx + \int_0^1 x^{-4/5}\,dx$$

But $$\int_{-1}^0 x^{-4/5}\,dx = \lim_{q \to 0} \int_{-1}^q x^{-4/5}\,dx = \lim_{q \to 0} 5x^{1/5}\Big]_{-1}^q = 5$$

And $$\int_0^1 x^{-4/5}\,dx = \lim_{p \to 0} \int_p^1 x^{-4/5}\,dx = \lim_{p \to 0} 5x^{1/5}\Big]_p^1 = 5$$

Therefore $$\int_{-1}^1 x^{-4/5}\,dx = 5 + 5 = 10$$

When, in any of the cases described, the defining limit exists, the improper integral in question is said to *converge*. Otherwise we say that the integral is *divergent* or that it does not exist.*

Example 7. Evaluate $\int_{-2}^{2} x^{-3}\,dx$ if it exists.

The integrand becomes infinite when $x = 0$. If the given integral exists, it has a value defined as

$$\int_{-2}^{2} x^{-3}\,dx = \int_{-2}^{0} x^{-3}\,dx + \int_{0}^{2} x^{-3}\,dx$$

But
$$\int_{-2}^{0} x^{-3}\,dx = \lim_{q\to 0} \int_{-2}^{q} x^{-3}\,dx = \lim_{q\to 0} \left(-\tfrac{1}{2}x^{-2}\right)\Big]_{-2}^{q} = -\infty$$

Therefore $\int_{-2}^{0} x^{-3}\,dx$ diverges, and we attach no value to it. Hence the original integral does not exist. It may be shown similarly that $\int_{0}^{2} x^{-3}\,dx$ diverges, taking the form $+\infty$. This is a second proof that the original integral does not exist. (We must not think that the original integral is then equal to $-\infty + \infty = 0$. For, informally, we see the fallacy of such a situation because infinite quantities are by no means necessarily equal to each other.)

A set of tables which includes many improper integrals useful to advanced workers is mentioned among the references.[5]

PROBLEMS

Evaluate the integrals of Probs. 1 to 13 if these integrals exist.

1. $\displaystyle\int_{1}^{\infty} \frac{dx}{x^2}$
2. $\displaystyle\int_{0}^{\infty} e^{-x}\,dx$
3. $\displaystyle\int_{0}^{\infty} \frac{dx}{(1+x)^2}$
4. $\displaystyle\int_{-\infty}^{0} e^{-x}\,dx$
5. $\displaystyle\int_{1}^{\infty} \frac{dx}{1+x^2}$
6. $\displaystyle\int_{1}^{4} \frac{x\,dx}{\sqrt{4-x}}$
7. $\displaystyle\int_{0}^{2} \frac{dx}{2-x}$
8. $\displaystyle\int_{0}^{4} \frac{dx}{\sqrt{x}}$
9. $\displaystyle\int_{0}^{1} \frac{dx}{x}$
10. $\displaystyle\int_{-1}^{1} x^{-2/3}\,dx$
11. $\displaystyle\int_{0}^{2} \frac{dx}{(x-1)^2}$
12. $\displaystyle\int_{-1}^{1} x^{-2}\,dx$
13. $\displaystyle\int_{-1}^{1} x^{-1/3}\,dx$

14. At time $t = 0$ an inductor of inductance L and of negligible resistance is connected through a resistance R to a source delivering a constant voltage V. Show that the energy stored in the magnetic field of the inductor approaches $W = V^2L/2R^2$ as t increases. (HINT: Use a procedure similar to that of Example 2.)

15. Referring to Example 2, find the total energy lost in R during the charging of the capacitor.

* In the case of certain integrals of the kind described under Case III, we may be able to ascribe to them values called *Cauchy principal values*, even though the integrals may not "exist" in the sense described above.[4] These values are named for the great French mathematician Augustin L. Cauchy (1789–1857).

16. The force between two magnetic poles of strengths m_1 and m_2, in a medium whose permeability is μ, is given by $\mathbf{F} = m_1 m_2/4\pi\mu s^2$, where s is the distance separating the poles. Find the work required to bring a north pole of strength 1 from an infinite distance to a distance S from a north pole of strength m. (NOTE: This result is principally of theoretical interest because of the impossibility of producing an isolated pole.)

17. Consider a straight cylindrical conductor of length s carrying a current I so that circular magnetic lines of force surround the conductor. It can be shown that the flux lying between distances r_1 and r_2 from the axis of the conductor is, under certain conditions,

$$\phi = \int_{r_1}^{r_2} \frac{2\mu\, sI}{r}\, dr \qquad \text{webers}$$

Does the total flux surrounding the conductor from $r = r_0$ (the radius of the conductor) and outwards approach any limit as $r \to \infty$, and if so, what limit? (If the result seems improbable, consider whether in practice there must not be a return conductor somewhere in order for the current to flow. Would not the field surrounding this return conductor certainly modify the above result?)

18. Strutt's *acoustic condition factor* for a studio (when noise is neglected) is

$$Q_1 = \frac{W/4\pi c D^2 + (W/V) \int_{t=0}^{1/16} \exp(-13.8t/T)\, dt}{(W/V) \int_{t=1/16}^{\infty} \exp(-13.8t/T)\, dt}$$

where c, D, T, V, and W are constants. Evaluate Q_1.

REFERENCES

1. F. L. GRIFFIN: "Mathematical Analysis: Higher Course," chaps. 3 and 4, Houghton Mifflin Company, Boston, 1927.
2. H. M. BACON: "Differential and Integral Calculus," 2d ed., chaps. 12 and 13, McGraw-Hill Book Company, Inc., New York, 1955.
3. G. CHRYSTAL: "Algebra," 5th ed., part I, chap. 8, reprint, Chelsea Publishing Co., New York, 1954.
4. C. R. WYLIE: "Calculus," pp. 179–180, McGraw-Hill Book Company, Inc., New York, 1953.
5. A. ERDELYI: "Tables of Integral Transforms," vols. I and II, McGraw-Hill Book Company, Inc., New York, 1954.

16

Double Integrals

In Chap. 14 we differentiated a function of two variables with respect to one of these variables, keeping the other constant. Here, we shall use the idea of *integrating* a function of two variables with respect to one of the variables, keeping the other variable constant.

16-1 Double integrals. Consider a function of two variables x and y, such as

$$x^2 + xy^2$$

The symbol

$$\int (x^2 + xy^2)\, dx \tag{1}$$

is taken to indicate the integral of the given function with respect to x, considering y as a constant. (Since the differential in (1) is dx, we know that it is with respect to x that the integration is to be performed. Sometimes this is further clarified symbolically, thus $\int^{(x)} (x^2 + xy^2)\, dx$.)

In this chapter we shall consider principally definite integrals. For example, let it be desired to evaluate the above integral between the limits $x = -1$ and $x = +2$. We get

$$\int_{x=-1}^{2} (x^2 + xy^2)\,dx = \left(\frac{x^3}{3} + \frac{x^2y^2}{2}\right)\Bigg]_{x=-1}^{2} = 3 + \frac{3}{2}y^2 \qquad (2)$$

We now propose a further problem. Let it be desired to integrate the result (2) with respect to y, keeping x constant. Let the desired limits be $y = 0$ and $y = 1$:

$$\int_{y=0}^{1} (3 + 3/2\,y^2)\,dy = (3y + 1/2\,y^3)\Big]_{y=0}^{1} = 7/2 \qquad (3)$$

Note that we have actually evaluated

$$\int_{y=0}^{1} \left[\int_{x=-1}^{2} (x^2 + xy^2)\,dx\right] dy \qquad (4)$$

An expression like (4) is usually written without the square brackets. And most workers follow the above *order* in writing the symbols, so it is often unnecessary to indicate, in conjunction with the integral signs, which variables are associated with the limits of integration. Thus, the expression (4) would appear*

$$\int_{0}^{1}\int_{-1}^{2} (x^2 + xy^2)\,dx\,dy \qquad (5)$$

This is called a *double integral.* Its meaning is clarified if we (mentally) insert the brackets as in (4), remembering that in integrating here with respect to any variable, we treat any other variables as constants.

Example. Evaluate $\int_{\pi/2}^{\pi}\int_{0}^{4} t^2 \sin 2\phi\,dt\,d\phi$.

In accordance with the foregoing, we are first required to find $\int_0^4 t^2 \sin 2\phi\,dt$, treating ϕ as a constant:

$$\int_0^4 t^2 \sin 2\phi\,dt = (1/3\,t^3 \sin 2\phi)\Big]_{t=0}^{4} = 64/3 \sin 2\phi$$

A second integration, this time with respect to ϕ and with t considered constant, completes the problem:

$$\int_{\pi/2}^{\pi}\int_0^4 (t^2 \sin 2\phi)\,dt\,d\phi = \int_{\pi/2}^{\pi} 64/3 \sin 2\phi\,d\phi = -\,32/3 \cos 2\phi\Big]_{\pi/2}^{\pi} = -\,64/3$$

PROBLEMS

Evaluate the following double integrals. Refer to tables of indefinite integrals wherever convenient.

* Another form in common use is $\int_0^1 dy \int_{-1}^{2} (x^2 - xy^2)\,dx$.

1. $\int_0^6 \int_0^1 x^2y\,dx\,dy$
2. $\int_2^3 \int_{-1}^5 xy\,dy\,dx$
3. $\int_0^8 \int_1^4 (x^2 - 3y^2)\,dy\,dx$
4. $\int_2^3 \int_2^4 (x^3 - y)\,dx\,dy$
5. $\int_1^4 \int_0^6 (xy^2 - x^2y)\,dx\,dy$
6. $\int_0^{\pi/2} \int_0^{\pi/2} \cos x \cos y\,dx\,dy$
7. $\int_0^2 \int_0^{\pi/2} (x^3 + \sin y)\,dy\,dx$
8. $\int_0^\pi \int_0^1 v \cos uv\,du\,dv$
9. $\int_0^\pi \int_0^2 x^2 \sin xy\,dy\,dx$
10. $\int_{-1}^1 \int_{-1}^1 e^{x-y}\,dx\,dy$
11. $\int_{-1}^1 \int_{-1}^1 ye^{xy}\,dx\,dy$
12. $\int_0^1 \int_0^{\pi/2} ye^{y \sin x} \cos x\,dx\,dy$

16-2 Areas expressed by double integrals. The majority of the area problems which you will encounter can be solved by simple integrations of the form $A = \int y\,dx$. But additional insight into the *double-integral* idea is provided if we apply that idea to the problem of calculating areas.

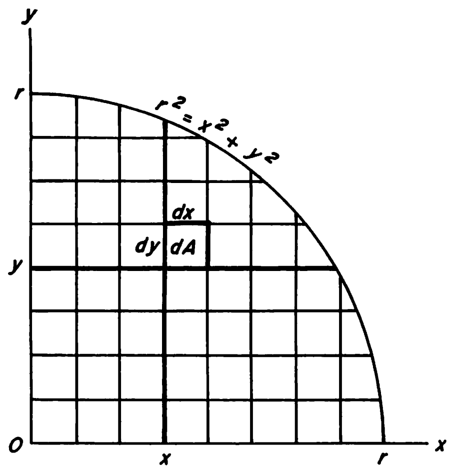


Fig. 16-1

a. Rectangular coordinates. In Fig. 16-1, let it be desired to find the area of a quarter or *quadrant* of the circle

$$x^2 + y^2 = r^2 \tag{6}$$

Consider the desired area A to be cut up by horizontal and vertical lines into small rectangles. Using, for brevity, the rough-and-ready idea of infinitesimal elements, we call the dimensions of each little rectangle dy and dx. Then we may refer to the area of each element as

$$dA = dy\,dx \tag{7}$$

Consider the rectangles as being so tiny that they may fit exactly into the curved figure, although we know this to be impossible. The area of a single vertical strip is then the *sum* of all the individual rectangles in that strip, each having an area $dy\,dx$. Let the symbol

$$\int_{y_1}^{y_2} dy\,dx \tag{8}$$

represent this sum, where y_1 and y_2 are the lower and the upper values of y in the strip. In general, y_1 and y_2 depend upon x. In the problem of Fig. 16-1 the strip extends from $y_1 = 0$ to $y_2 = (r^2 - x^2)^{1/2}$.

The total desired area A is the sum of all the vertical strips of the form (8). Let the symbol

$$A = \int_{x_1}^{x_2} \int_{y_1}^{y_2} dy\, dx \tag{9}$$

indicate this sum, where x_1 and x_2 are the limits over which x varies to include the desired area. In the problem of Fig. 16-1 $x_1 = 0$ and $x_2 = r$. The area of the quadrant is then

$$A = \int_0^r \int_0^{\sqrt{r^2-x^2}} dy\, dx = \int_0^r \sqrt{r^2 - x^2}\, dx \tag{10}$$

The latter integral is readily evaluated through the expedient of factoring from the radical the quantity x^2 (or, really, x):

$$A = \int_0^r \sqrt{r^2 - x^2}\, dx = \int_0^r x \sqrt{r^2x^{-2} - 1}\, dx$$

We now apply integration by parts, letting $u = (r^2x^{-2} - 1)^{1/2}$, $dv = x\, dx$, so that $du = -r^2x^{-3}(r^2x^{-2} - 1)^{-1/2}\, dx$, and $v = x^2/2$:

$$A = \int_0^r x \sqrt{r^2x^{-2} - 1}\, dx = \frac{x^2}{2}(r^2x^{-2} - 1)^{1/2}\Big]_0^r + \frac{r^2}{2}\int_0^r \frac{dx}{x(r^2x^{-2} - 1)^{1/2}}$$

In each of the two right-hand terms we may now divide the quantity outside the parentheses by x and multiply the quantity inside the parentheses by x^2:

$$\begin{aligned} A &= \frac{x}{2}(r^2 - x^2)^{1/2}\Big]_0^r + \frac{r^2}{2}\int_0^r \frac{dx}{(r^2 - x^2)^{1/2}} \\ &= \left[\frac{x}{2}(r^2 - x^2)^{1/2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}\right]_0^r = \frac{1}{4}\pi r^2 \end{aligned}$$

(Thus the area of an entire circle of radius r is πr^2.)

Two important comments follow:

1. The area could as well have been figured by first summing the small rectangles along a *horizontal* strip, then summing all such strips according to the integral

$$A = \int_0^r \int_0^{\sqrt{r^2-y^2}} dx\, dy$$

2. In either order of integration it is essential that a *variable* strip be used as the basis of the first integration. For instance, in (10) it would *not* do to perform the first integration simply from $y = 0$ to $y = r$, for this would involve only a fixed strip along the edge of the desired area. What we need is the area of a *general* strip whose height varies with x, so that *all* such strips will be included in the second integration.

Example. Find by double integration the area to the right of the origin and between the parabola $y = x^2$ and the straight line $y = 2x$.

The desired area is shown in Fig. 16-2. The area of a single vertical strip is

$$\int_{y=x^2}^{2x} dy\,dx$$

and the sum of all such strips is the total area:

$$A = \int_{x=0}^{2} \int_{y=x^2}^{2x} dy\,dx = \int_0^2 (2x - x^2)\,dx = 4/3 \text{ area units}$$

(*b*) *Polar coordinates.* Most area problems involving polar coordinates are to be solved by the formula

$$A = 1/2 \int_{\theta=a}^{b} r^2\,d\theta$$

(Sec. 11-15). The following treatment illustrates the application of double integrals to such problems.

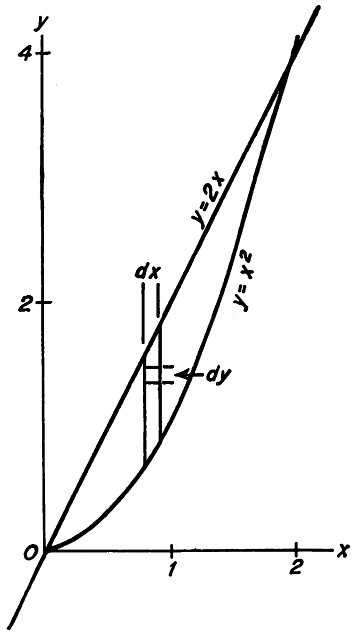


Fig. 16-2

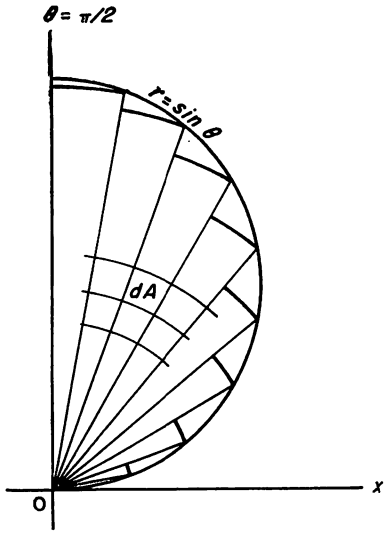


Fig. 16-3

Figure 16-3 illustrates the curve $r = \sin\theta$. Let it be desired to find the area enclosed by this curve, the polar axis $\theta = 0$, and the line $\theta = \pi/2$. Consider the area in question to be cut up into small sections by means of radii drawn from the pole O and by means of arcs drawn with O as center. Then the desired area can be taken as the sum of all these little

sections. (Again, we are thinking of the small sections as being so tiny that they fit exactly into the curved area we are considering.)

Figure 16-4 shows one small section alone. Its distance from O is indicated by r. Let $d\theta$ indicate the "differential angle" between radii. Then the length of the arc along the end of the section is $r\,d\theta$. Since the sections are tiny, we take the width of the section as being constant ($= r\,d\theta$) along its length dr. Thus we have, approximately, a little "rectangle" whose area is $r\,dr\,d\theta$. The sum of all such rectangles along a given radius can be represented by

$$\int_{r_1}^{r_2} r\,dr\,d\theta \qquad (11)$$

where r_1 and r_2 are the limits between which r varies in describing the area A. In the case of Fig. 16-3, $r_1 = 0$, and $r_2 = \sin\theta$.

The sum of all the rows of rectangles, each row being of the form (11), can be represented by the total area:

$$A = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} r\,dr\,d\theta \qquad \textbf{(12)}$$

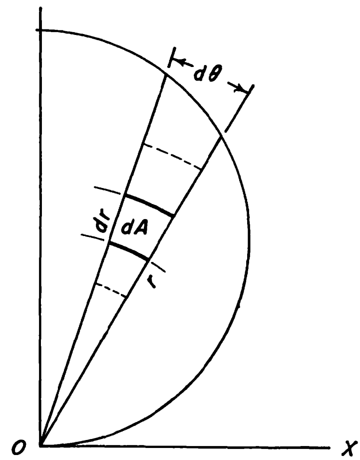


Fig. 16-4

where θ_1 and θ_2 are the limits between which the angle θ varies in describing the area A. In the problem of Fig. 16-3 the total area is

$$A = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\sin\theta} r\,dr\,d\theta$$

The inner integral is equal to

$$\int_0^{\sin\theta} r\,dr = \frac{r^2}{2}\bigg]_0^{\sin\theta} = \frac{\sin^2\theta}{2}$$

Then

$$A = \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{1}{2}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\bigg]_0^{\pi/2} = \frac{\pi}{8}\text{ area units}$$

As in the case of rectangular coordinates, the order of integration can be reversed, using if desired $A = \iint r\,d\theta\,dr$. And here, too, in the first integration, we must use a *general* row of area elements, *not* simply a row along one edge of the desired area, for example.

PROBLEMS

Set up and solve the following problem, using double integration. Use rectangular or polar coordinates, as convenient.

1. A cross section along the axis of a certain antenna reflector is described by the equation $y^2 = 48x$. Find the area of this section from $x = 0$ to $x = 12$.

2. The cardioid directional pattern of a certain microphone is approximated by a graph $r = a(1 + \cos\theta)$ where a is a constant. Find the area of this graph.

3. The height h feet of an antenna wire above a flat earth at a horizontal distance x feet from the center point is $h = 50 \cosh 0.02x - 10$. Find the area described by the wire, its two supporting poles 100 feet apart, and the earth.

4. A tapered open-wire transmission line is used to match two unequal impedances. The distance y of either conductor from a center line varies with distance x along the center line according to $y = ae^{bx}$ centimeters, where a and b are constants. Find the area between the conductors from $x = 0$ to $x = k$ centimeters.

5. Find the area of one rotor plate of a straight-line-wavelength variable capacitor if its radius varies with angle θ as $r = [A(B\theta + C)/D + E]^{1/2}$ where A, B, C, D, and E are constants. Here θ varies from $\theta = 0$ to $\theta = \theta_{max}$ radians.

6. Same as Prob. 5, except for a straight-line-frequency capacitor, with $r = [K/M(N - \theta)^3 + P]^{1/2}$, where K, M, N, and P are constants. Let θ vary from $\theta = 0$ to $\theta = \theta_{max}$ radians.

7. Find the area common to the cardioid antenna pattern $\mathbf{E} = 1 + \cos\theta$ and the circular pattern $\mathbf{E} = 1 + \sqrt{2}/2$.

16-3 Volumes expressed by double integrals. (*a*) *Cartesian coordinates.* Let it be desired to find the volume V of a solid (Fig. 16-5)

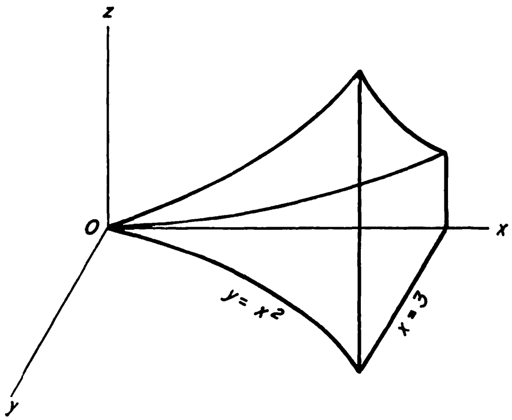


Fig. 16-5

whose upper surface is described by

$$z = x^2 + y^2$$

and whose base is bounded by the x and y axes, the line $x = 3$, and the curve $y = x^2$. In Chap. 10 we found that the volume of a solid is

$$V = \int_{x_1}^{x_2} A\,dx \tag{13}$$

where A is a general cross-sectional area expressed in terms of x. But we must evaluate A as a function of x, so that the integration (13) may be performed. The area A under a curve (such as the upper

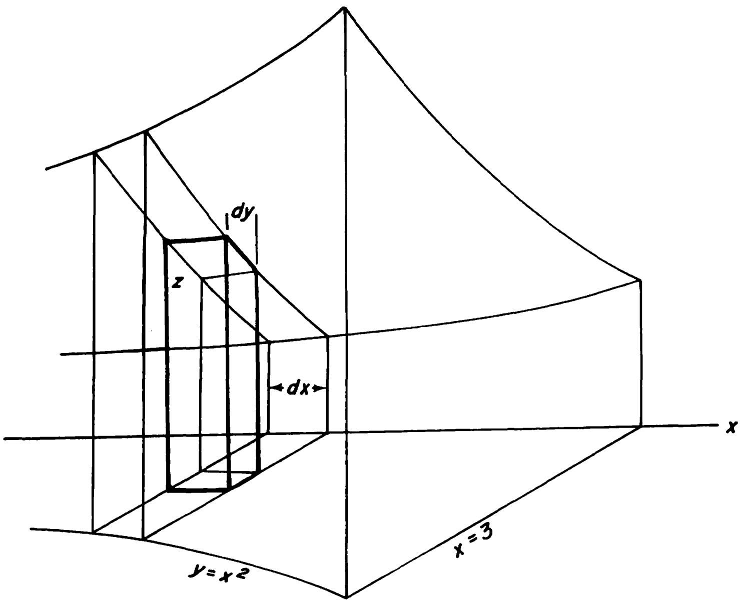


Fig. 16-6

boundary of A) can be evaluated as

$$A = \int_{y_1}^{y_2} z\, dy \tag{14}$$

Substituting (14) in (13),

$$V = \int_{x_1}^{x_2} \int_{y_1}^{y_2} z\, dy\, dx \tag{15}$$

In the given problem, this becomes

$$V = \int_0^3 \int_0^{x^2} (x^2 + y^2)\, dy\, dx = 152.7^+ \text{ volume units}$$

From the standpoint of infinitesimal elements, the volume V can be considered as made up of columns, each having a base area $dy\, dx$ and height z (Fig. 16-6). Then let (15) represent the sum of all such columns, totaled first along a general section having a fixed value of x to give the volume $A\, dx$ of one "slice." The total of all such slices, as x varies, gives V.

The order of integration in (15) can be reversed:

$$V = \int_{y_1}^{y_2} \int_{x_1}^{x_2} z\,dx\,dy$$

Using either order of integration, we must evaluate in the first integration a *general* slice, not merely one along an edge of the desired volume, for example.

b. Cylindrical coordinates. Where the height z of the surface of a solid is expressed in cylindrical coordinates (Fig. 16-7) the element of volume becomes a column whose base area is $r\,dr\,d\theta$ and whose height is z.

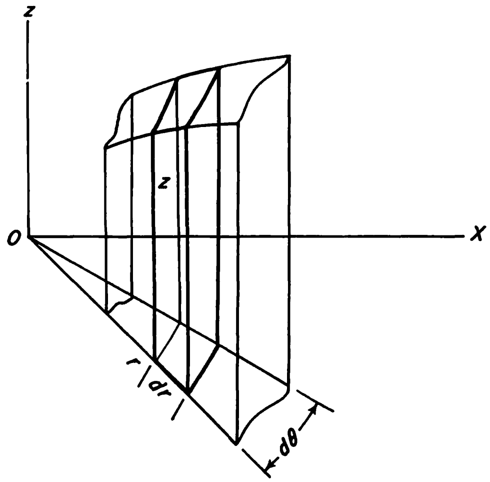


Fig. 16-7

The volume of the element is therefore $rz\,dr\,d\theta$. The sum of these elements taken along one radius can be represented by

$$\int_{r_1}^{r_2} rz\,dr\,d\theta$$

(If r_1 happens to equal zero, this forms a narrow "piece of pie".) The total of all such pieces is then the volume of the solid:

$$V = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} rz\,dr\,d\theta \tag{16}$$

QUESTIONS

1. A function $f(x,y)$ of two independent variables x and y is to be integrated, first with respect to x (treating y as a constant) then with respect to y (treating x as a constant). Write two common symbols for this *double integral.*

2. Give formulas for double integrals expressing areas of irregular plane surfaces (a) in rectangular coordinates and (b) in polar coordinates.

3. Give formulas for double integrals expressing volumes of solids (*a*) in cartesian coordinates and (*b*) in cylindrical coordinates.

PROBLEMS

Set up and solve the following problems, using double integration. Use cartesian or cylindrical coordinates, as convenient.

1. In order to estimate public-address power requirements for a room of irregular shape, it was desired to find the volume of the room. Find this volume, if the

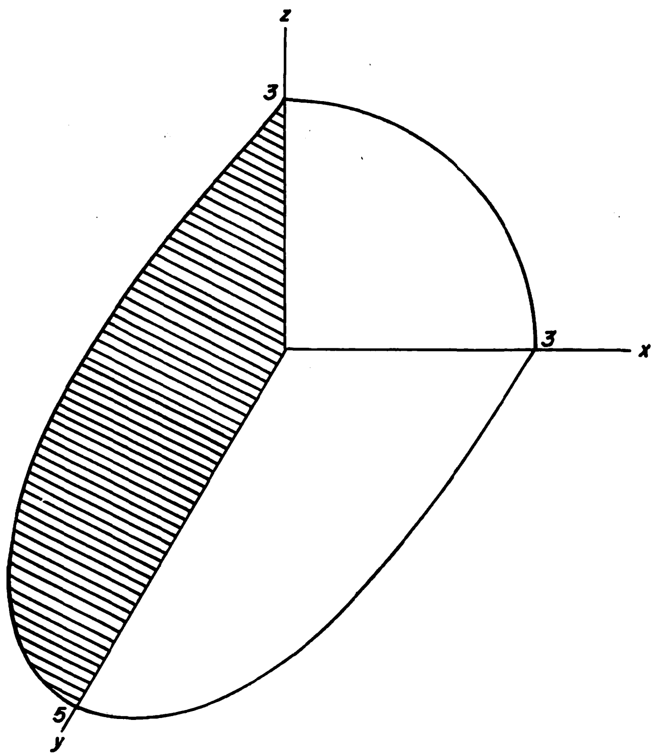


Fig. 16-8

height of the room varied as $z = 0.001x^2 - 0.4y + 40$ feet. One side wall lay along the x axis, and one end wall lay along the y axis. The end walls were parallel and 100 feet apart. The fourth wall curved, its distance from the x axis being $y = 48 - 0.0018x^2$ feet.

2. The surface of an egg-shaped antenna insulator has the equation

$$\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{9} = 1$$

where x, y, and z are in centimeters. Find the volume of the insulator, neglecting the presence of holes and grooves for wires. [HINT: The volume is eight times that of the portion shown in Fig. 16-8. From the given equation, $z = 3(1 - x^2/9 - y^2/25)^{1/2}$.

A general slice in the y direction extends from $y_1 = 0$ to $y_2 = 5(1 - x^2/9)^{1/2}$, found by letting $z = 0$ in the given equation. Thus

$$V = 8 \int_0^3 \int_0^{5\sqrt{1-x^2/9}} 3\left(1 - \frac{x^2}{9} - \frac{y^2}{25}\right)^{1/2} dy\,dx$$

The inner integral is evaluated in the manner of (10).]

3. A loudspeaker horn is square in cross section. The height of the upper surface above the central axis varies with distance x along the axis as $z = me^{nx}$. (Similarly, the distance from the axis to a side of the horn is $y = me^{nx}$.) Find the volume of the horn from $x = 0$ to $x = p$. HINT: Show that

$$V = 4 \int_0^p \int_0^{me^{nx}} me^{nx}\,dy\,dx$$

4. Find the volume contained in an antenna reflector whose surface has the equation $kx = y^2 + z^2$, from $x = 0$ to $x = a$.

5. Show that the volume of a sphere is $V = 4\pi r^3/3$.

6. Two waveguides are circular in cross section, and each has an inner diameter of 10 centimeters. The axes of the guides intersect at right angles. Find the volume common to the two waveguides.

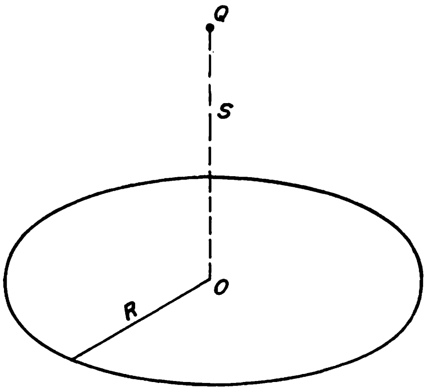


Fig. 16-9

16-4 Conclusion.

The basic idea of the double integral is capable of many applications and extensions which are of use in electricity. For brevity, we shall only mention some examples.

The double integral is useful in finding amounts of *forces* (attractions and repulsions) in certain problems in electricity and magnetism.

Example 1. A horizontal circular disk of radius R (Fig. 16-9) has upon its surface a uniformly distributed charge of k coulombs per square meter. Find the repelling force upon a charge of Q coulombs and of similar polarity situated at a point S meters vertically above the center O of the disk.

Using cylindrical coordinates, let us break up the plate into elements, each having an area $r\,dr\,d\theta$, where r is the distance of the element from the center of the plate. The charge carried by each element, for instance, one located at P, is then

$$q = kr\,dr\,d\theta \qquad \text{coulombs} \tag{17}$$

The force acting between this charge and the external charge is, by the law of Coulomb,

$$d\mathbf{F} = -\frac{Qq}{4\pi\epsilon s^2} \qquad \text{newtons} \tag{18}$$

Here ϵ is the permittivity of the medium and s is the distance separating the

element under consideration from the external charge. Substituting (17) in (18),

$$d\mathbf{F} = -\frac{kQr\,dr\,d\theta}{4\pi\epsilon s^2} \qquad \text{newtons} \tag{19}$$

This force acts along a line connecting P with Q. It can be resolved into a vertical component

$$d\mathbf{F}_z = -d\mathbf{F}\cos\angle OPQ = -\frac{kQr\,dr\,d\theta}{4\pi\epsilon s^2}\cos\angle OPQ \qquad \text{newtons} \tag{20}$$

and a horizontal component

$$d\mathbf{F}_r = -d\mathbf{F}\sin\angle OPQ = -\frac{kQr\,dr\,d\theta}{4\pi\epsilon s^2}\sin\angle OPQ \qquad \text{newtons} \tag{21}$$

[Since Q is directly above the center of the disk in this problem, the horizontal components of the forces acting upon the various elements will balance out, and these forces of the form (21) may be neglected.] We note that $s^2 = S^2 + r^2$ and that $\cos\angle OPQ = S/(S^2 + r^2)^{1/2}$. Substituting these values into (20),

$$d\mathbf{F}_z = -\frac{kSQr\,dr\,d\theta}{4\pi\epsilon(S^2 + r^2)^{3/2}} \qquad \text{newtons} \tag{22}$$

The "sum" of these forces due to charges upon elements along one radius of the disk is

$$-\frac{kSQ}{4\pi\epsilon}\int_0^R \frac{r}{(S^2 + r^2)^{3/2}}\,dr\,d\theta$$

And the total force is

$$\mathbf{F} = -\frac{kSQ}{4\pi\epsilon}\int_0^{2\pi}\int_0^R \frac{r\,dr\,d\theta}{(S^2 + r^2)^{3/2}} = \frac{kQ}{2\epsilon}\,\frac{S - (S^2 + R^2)^{1/2}}{(S^2 + R^2)^{1/2}} \qquad \text{newtons} \tag{23}$$

The total effect of a quantity which varies in a known manner over an irregular area is often found by a double integral.

Example 2. A thin, flat insulating plate has the form of that portion of the hyperbola $y = 1/x$ lying above the x axis from $x = 1$ to $x = 10$. Find the mass of the plate if its surface density at any point P is $D = 5/x$.

The mass of any element of the plate is $dM = D\,dy\,dx = (5/x)\,dy\,dx$. The "sum" of these masses along any straight line in the y direction is $5\int_0^{1/x}(1/x)\,dy\,dx$. The mass of the plate is then

$$M = 5\int_1^{10}\int_0^{1/x}\frac{1}{x}\,dy\,dx = 4.5 \text{ units}$$

Example 3. A plate has the form of an ellipse defined by

$$r^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} \qquad \text{centimeters}$$

A hole 1 centimeter in radius is centered at the origin. An electric charge is distributed over the plate in such a way that at any point $P(r,\theta)$ the density of the charge is $D = p/r^4$ coulombs per square centimeter. Find the total charge on the plate if $a = 2$ centimeters, $b = 3$ centimeters, and $p = 10^{-6}$.

The charge on any area element $r\,dr\,d\theta$ is $Dr\,dr\,d\theta = p/r^3\,dr\,d\theta$. The "sum" of these charges along any radius is

$$p \int_1^{\sqrt{a^2b^2/(a^2\sin^2\theta+b^2\cos^2\theta)}} \frac{dr\,d\theta}{r^3}$$

The total charge on the plate is then

$$Q = p \int_0^{2\pi} \int_1^{\sqrt{a^2b^2/(a^2\sin^2\theta+b^2\cos^2\theta)}} \frac{dr\,d\theta}{r^3} = \frac{59\pi}{72} \times 10^{-6}\text{ coulomb}$$
$$= 2.574 \text{ microcoulombs}$$

The ideas already used in the double integral may be extended to further integrations. For example, a function $f(x,y,z)$ of three independent variables x, y, and z may be integrated first with respect to x, then with respect to y, and finally with respect to z—in each integration, of course, treating as constants any variables other than the one with respect to which we are integrating. This result is called a *triple integral.* The notation used is as would be expected:

$$\int_{z_1}^{z_2} \left\{ \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} f(x,y,z)\,dx \right] dy \right\} dz$$

would be written

$$\int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz \qquad \text{or} \qquad \int_{z_1}^{z_2} dz \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f(x,y,z)\,dx$$

Sometimes even further integrations are called for. In some antenna studies, for instance, we may find sextuple integrals useful.

Part Five

INFINITE SERIES

17

Maclaurin's Series

For many functions, it is possible to find an endless series of simple terms, the "sum" of which will represent the function. Such a set of terms is called an *infinite series.* It might at first be thought that such a series would be too cumbersome for practical use. Quite the contrary is true, however. It is found that infinite series are of great value in very practical ways.

In applying calculus to the field of electricity, we can use infinite series in such ways as these:

1. Defining and illustrating some useful functions which are not otherwise obtainable.

2. Illustrating methods of preparing tables of functions, such as exponentials, sines, etc.; or even in calculating the values of such functions if necessary.

17-1 Sequences. Let us consider a form called a *sequence.* This consists simply of a set of quantities*

$$u_0,\ u_1,\ u_2,\ u_3,\ \ldots,\ u_n \tag{1}$$

* Sometimes it is convenient to begin the sequence with u_1, writing: $u_1, u_2, u_3, \ldots, u_n$.

The terms of (1) are assumed to be related to each other through some *definite law.* The set of dots is a convenient way of allowing for the writing of as many terms as may be desired, each term being derived from the preceding ones by the same law.

One of our earliest mathematical experiences is that of counting, or naming the successive terms of the sequence

$$1, 2, 3, 4, \ldots \tag{2}$$

This is an example of an *arithmetical* sequence, in which each term is separated from the previous one in value by a common difference d. In (2), we have let $d = 1$.

Another familiar form is that of the *geometrical* sequence, in which each term is the product of the previous one times a common ratio r. An example is

$$1, 2, 4, 8, 16, \ldots \tag{3}$$

In (3), we made $r = 2$, and the first term was taken as 1.

A geometrical sequence could be used, for example, to show the way in which the intensity of a signal changes as it is sent through successive stages of amplification. If the input signal was V volts, and if each stage gave a gain of k times, then the various stages would provide output voltages

$$kV, k^2V, k^3V, k^4V, \ldots$$

a geometrical sequence where $r = k$.

PROBLEMS

1. Write three further terms for (*a*) the sequence (2) and (*b*) the sequence (3).

2. Write four terms of a sequence showing how a signal of 0.002 volts is amplified through an amplifier having a gain per stage of 20.

3. Write four terms of a sequence showing how a 5-microampere output from a photoelectric cathode is increased by an electron multiplier, each plate of which provides an output current four times its input.

4. Write five terms of a sequence illustrating the loss in signal power as an input of 1 watt is successively reduced by attenuating pads, each of which gives an output of one-fourth its input power.

5. In a *distributed amplifier* (or *transmission-line* amplifier) an input signal voltage V is introduced into a circuit in which tubes are inserted at intervals. Each tube in turn adds an amount of signal voltage equal to some fraction a times the original voltage V. Write a sequence showing the values of voltage available at the output of each of four tubes in such an amplifier.

17-2 Power series. Let us look at an example of *finite* series (a series having only a limited number of terms). Consider a hypothetical diode having a characteristic curve of plate current i_b versus plate voltage v_b

as shown in Fig. 17-1*a*. It will be noted that this curve does not follow any simple formula such as $i_b = kv_b^2$; however, in Fig. 17-1*b* we see that it can be closely approximated over the range shown by the *sum* of three simple terms:

$$20v_b + 10v_b^2 + 0.1v_b^3 \tag{4}$$

If (4) did not give the current function with sufficient accuracy, a better

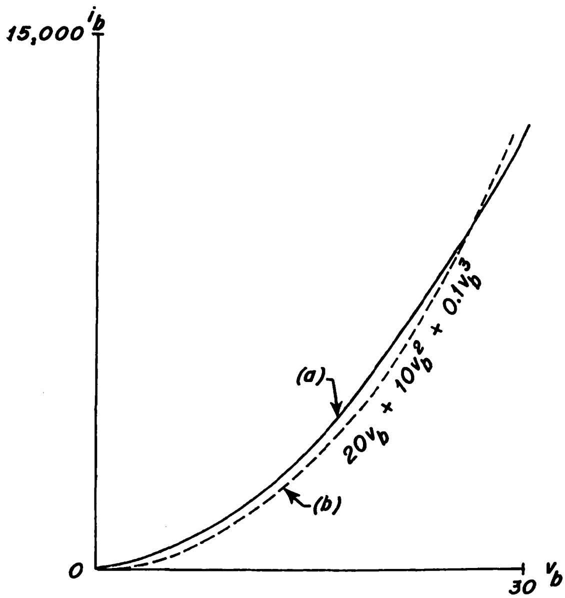


Fig. 17-1

approximation could be had in many cases by supplying further terms made up of higher powers of v_b.

Note that (4) is the *sum* of a set of algebraic *power* functions. A general form for such a sum follows, allowing for any desired number of terms:

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots \tag{5}$$

When each term of (5) has a numerical coefficient A, B, etc., derived from the previous terms according to some *definite* law, we refer to (5) as a *power series*. It is to be observed that a *series* is the sum of the terms of a *sequence*.

17-3 Infinite series. An infinite series is an expression of the form

$$u_0 + u_1 + u_2 + u_3 + \cdots + u_n + \cdots \tag{6}$$

allowing for an *endless* number of terms. Some of the terms may be

negative. The successive terms are derived from the previous ones in some definite manner.

An infinite series of *constant terms* might be, for instance, one made up of the terms of a geometrical sequence:

$$1 + 2 + 4 + 8 + \cdots \tag{7}$$

where $r = 2$; or

$$8 + 4 + 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots \tag{8}$$

where $r = \frac{1}{2}$.

An example of an *infinite power series* follows. Consider the function

$$y = \frac{1}{1 + x} \tag{9}$$

If we carry out the division indicated in the right member according to the customary procedure of algebra, we get

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots \tag{10}$$

and so on endlessly. Here, we have a series of the form (5), with the coefficients being equal alternately to $+1$ and -1. This series is an example of an infinite series of *functions*. In a later chapter we shall study infinite series of trigonometric functions, rather than power functions.

Strictly speaking, a series like (7), (8), or (10) does not have a *sum*. For no matter how many terms we might add together, the total which we got could always be modified at least slightly by taking further terms into consideration.

But it is entirely possible that the sum of the values of the terms of an infinite series might approach a limit as we take in a greater and greater number of terms. If there is such a limit, we can *define* it as the value or "sum" of the series:

➢ If the sum of n terms of a series approaches a limit S as n increases without bound, then the series is said to be convergent, and S is the value assigned to the series. A series which is not convergent is said to be divergent.

We assign no value here to a divergent series. (Some applications, of an advanced nature, have been found for certain nonconvergent series.)

Let us consider (7) and (8) in the light of the above definition. The series (7) is clearly divergent; in fact, the sum of its terms increases at a greater and greater rate as the series progresses. Looking at (8), we see that the terms become smaller and smaller. This leads us to think

that a limit *might* exist for this series. But a more complete study would show that this fact alone is not enough to prove convergence. Tests for convergence are given in formal calculus texts.[1] Here we shall only say that a geometrical series converges if r is numerically smaller than 1, as in (8); otherwise it diverges.

The power series (10) is seen to be a geometrical series with $r = -x$. Thus it converges for values of x between -1 and $+1$, and it may be used to represent the quantity $1/(1 + x)$ for values of x in that range.

Example 1. Write an infinite power series representing the function $1/(1 - x)$.

Carrying out the division $1 \div (1 - x)$, we get

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometrical series with $r = x$, and thus converges for values of x between -1 and $+1$.

Example 2. One way in which a feedback amplifier may be connected is shown in Fig. 17-2. Let A represent the gain of the amplifier in the absence of feedback,

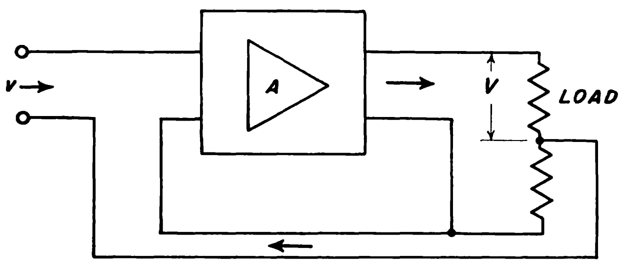


Fig. 17-2

and assume that no phase shifts are present. In this particular hookup, a fraction β is subtracted from the total amplifier output voltage and fed back into the input circuit. Find the effect upon the gain.

If an input signal voltage v is supplied to the amplifier, the resulting total output voltage will be Av. Of this total, a voltage βAv is fed back to the input, leaving a useful output voltage

$$Av - \beta Av = Av(1 - \beta)$$

But the fed-back voltage βAv will now be amplified, producing an additional output voltage βA^2v. Of this, a voltage β^2A^2v is fed back, leaving a voltage $\beta A^2v(1 - \beta)$ as an additional contribution to the useful output. This process may be pictured as going on indefinitely. The operation is one of cause and effect, with essentially no time delay involved in the successive operations. The useful output voltage V then becomes the sum of an infinite series of terms

$$\begin{aligned} V &= Av(1 - \beta) + \beta A^2v(1 - \beta) + \beta^2A^3v(1 - \beta) + \cdots \\ &= Av(1 - \beta)(1 + A\beta + A^2\beta^2 + A^3\beta^3 + \cdots) \end{aligned}$$

By Example 1, the series in the right-hand factor of this expression converges if $A\beta$ is less than 1. It has the value $1/(1 - A\beta)$. This makes

$$V = Av \frac{1 - \beta}{1 - A\beta}$$

so that the gain of the amplifier with feedback becomes

$$A_{fb} = A \frac{1 - \beta}{1 - A\beta}$$

If the voltage fed back is in phase with v, a condition of *regeneration* will prevail, with A_{fb} becoming greater than A. In the limiting case, as $A\beta$ is made close to 1 by increasing β, the value of A_{fb} increases without bound. Inspecting the equation for the output voltage V, we find that sustained oscillations occur if $A\beta$ approaches 1, since this permits V to have a value other than zero even if v is made vanishingly small.

When the voltage fed back is opposite in phase to v, as indicated by a negative value of β, a condition of *negative feedback* (or *inverse* feedback) applies. The gain then decreases as β is made more negative, taking the limiting value 1 when β increases without limit negatively.

17-4 Calculation of logarithms. In our studies of integration and differentiation, we found that the integral (or derivative) of the sum of a finite number of terms is equal to the sum of the integrals (or derivatives) of the individual terms. In higher courses it is proved valid to integrate (or differentiate) a convergent *infinite* power series term by term. If we integrate each term of (10), we get

$$\ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{11}$$

This series converges if x is greater than -1 but equal to, or less than, $+1$. By means of (11), we can obtain the logarithms of numbers close to 1. The series does not serve, of course, for values of x outside its range of convergence.

Example. In (11), set $x = 0.01$, getting the logarithm of 1.01. This gives

$$\ln 1.01 = 0.01 - \frac{(0.01)^2}{2} + \cdots = 0.00995$$

An important point is that series may converge so slowly as to be of limited practical value. In other words, if for a given value of x, many terms of a series are required to approach closely to the value of the limit, then the series is inconvenient or perhaps useless for practical applications for such values of x.

For small values of x, like 0.01, the above series converges in only two terms to the values given in the five-place tables. If, however, we should try to evaluate ln 1.99 by this method, the sum of many terms would be needed for five-place accuracy. This illustrates the problem of a slowly converging series.

The series (11) is useful, then, in getting logarithms of numbers near 1. Methods of combining series may be used to get logarithms of other numbers.[2]

17-5 Maclaurin's series. The fact that we have been able to find an infinite power series representing the logarithmic function makes us wonder about the possibility of getting similar series for other common functions. Although the method to be shown is not just like that used to get the series for the logarithm, it is nevertheless often easy. Assume, for instance, that we want a power series representing $\sin x$.

This series, if it exists, will be of the form

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots \tag{5}$$

Here $f(x)$ represents $\sin x$. Our problem consists of finding the values of the coefficients A, B, C, etc., so that we can write the series.

First, *assuming* that a power series exists for the sine function, we write it in the same form as (5). Then we differentiate several times:

$$\begin{aligned}
\sin x &= A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \cdots & (12)\\
\cos x &= B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \cdots & (13)\\
-\sin x &= 2C + 6Dx + 12Ex^2 + 20Fx^3 + \cdots & (14)\\
-\cos x &= 6D + 24Ex + 60Fx^2 + \cdots & (15)\\
\sin x &= 24E + 120Fx + \cdots & (16)\\
\cos x &= 120F + \cdots & (17)
\end{aligned}$$

It is now necessary to assume that the above equations are true when $x = 0$. If they are, for the function $f(x)$ involved, we can substitute zero for x in each of the above equations, getting

$$\begin{aligned}
\sin 0 &= A & \text{or} \quad A &= 0 & (12a)\\
\cos 0 &= B & \text{or} \quad B &= 1 & (13a)\\
-\sin 0 &= 2C & \text{or} \quad C &= 0 & (14a)\\
-\cos 0 &= 6D & \text{or} \quad D &= -1/6 & (15a)\\
\sin 0 &= 24E & \text{or} \quad E &= 0 & (16a)\\
\cos 0 &= 120F & \text{or} \quad F &= 1/120 & (17a)
\end{aligned}$$

We have now evaluated the needed coefficients. Substituting these results into (12),

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \tag{18}$$

This series enables us to calculate the sine of an angle simply by substituting the value of the angle, in radians, into the series.

Carefully observe the way in which we got this result. First, we wrote the assumed series, but with letters representing the unknown coefficients. Second, we differentiated the result several times. Third, we successively substituted zero for the quantity x in the original series and its derivatives. This permitted us to evaluate the coefficients.

Series of this kind were discovered about 1740 by Colin Maclaurin of Scotland, and hence they are called *Maclaurin's series.* Actually, as Maclaurin recognized, they are a special case of the Taylor's series, treated in the next chapter.

17-6 Factorial quantities. The writing of a Maclaurin's series, and certain other expressions, is simplified by a notation called the *factorial notation:*

➤ Letting n be a positive whole number, we call the product of all the whole numbers from 1 to n, inclusive, factorial n and give it the symbol $n!$. (An older symbol is $\lfloor\underline{n}$.)

Some of the factorials are

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \\ 3! &= 6 \\ 4! &= 24 \\ 5! &= 120 \end{aligned}$$

Considerations which arise in more advanced courses cause us to give the value 1 to $0!$.

Inspecting (18), we see that it could be written

➤ $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{18a}$$

If we look over the procedure used in getting this series for $\sin x$, we see from the very manner of its derivation that

1. The terms of the series are alternately plus and minus.
2. The terms involve odd powers of x.
3. The denominator of each term is the factorial of the exponent of the power of x in the numerator of that term.

Having once established this pattern, we need only write the new terms according to the above rules in order to get further terms of the series and hence greater accuracy in any calculations based on the series. For any function having a Maclaurin's series, whether it is $\sin x$, $\cos x$, e^x, or any other, there will be *some* pattern by which the successive terms are established. It is only necessary to determine what this pattern is.

Tables of factorials, their reciprocals, and sometimes their logarithms are included in books of mathematical tables.

Example 1. Obtain a Maclaurin's series for e^x.
This series will have the form

$$e^x = A + Bx + Cx^2 + Dx^3 + \cdots$$

Differentiating repeatedly,

$$\begin{aligned} e^x &= B + 2Cx + 3Dx^2 + \cdots \\ e^x &= 2C + 6Dx + \cdots \\ e^x &= 6D + \cdots \end{aligned}$$

Setting $x = 0$,

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= \frac{1}{2} = \frac{1}{2!} \\ D &= \frac{1}{6} = \frac{1}{3!}, \text{ etc.} \end{aligned}$$

so that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{19}$$

Example 2. Find the form approached by the current wave in an inductance to which a fixed dc voltage is suddenly applied as the amount of resistance is reduced.

In Chap. 12, we found that, when a dc voltage V is suddenly applied to an RL series circuit, the current wave has the form $i = (V/R)(1 - e^{-Rt/L})$. Applying the result of Example 1, above,

$$i = \frac{V}{R}\left[1 - \left(1 - \frac{Rt}{L} + \frac{R^2t^2}{L^2 2!} - \frac{R^3t^3}{L^3 3!} + \cdots\right)\right]$$

If now R in this expression be made to approach zero, then the terms having higher powers of R will become extremely small. Neglecting, then, all except the first two terms of the series, we have the linear expression

$$i = \frac{Vt}{L}$$

for the limiting form approached by the current wave as R decreases.

An application of this information would be in a television deflecting coil having small resistance. Approximately a *rectangular* wave of voltage applied to the coil would cause the desired sawtooth deflecting current. In practice, the result given is only approximate, since the circuit will always have some resistance.

17-7 Maclaurin's series—general form.* For any function $f(x)$ having a Maclaurin's series, the series must be

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots \tag{5}$$

Differentiating repeatedly,

$$f'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 + \cdots \tag{20}$$
$$f''(x) = 2C + 6Dx + 12Ex^2 + \cdots \tag{21}$$
$$f'''(x) = 6D + 24Ex + \cdots \tag{22}$$
$$f^{\text{iv}}(x) = 24E + \cdots \tag{23}$$

etc. If we now let x take the value zero, then in (5) the symbol $f(x)$ can be changed to $f(0)$, indicating the particular value taken by $f(x)$ when $x = 0$. Also, in (20), if we let $x = 0$, then $f'(x)$ will be the value of the derivative of $f(x)$ at the point $x = 0$, and this can be written $f'(0)$. Similarly for (21) to (23), so that if we let $x = 0$ to evaluate the various coefficients, we get

$$f(0) = A \qquad A = f(0) \tag{5a}$$
$$f'(0) = B \qquad B = f'(0) \tag{20a}$$
$$f''(0) = 2C \qquad C = \frac{f''(0)}{2!} \tag{21a}$$
$$f'''(0) = 6D \qquad D = \frac{f'''(0)}{3!} \tag{22a}$$
$$f^{\text{iv}}(0) = 24E \qquad E = \frac{f^{\text{iv}}(0)}{4!} \qquad \text{etc.} \tag{23a}$$

Making the above substitutions in the original series (5),

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{\text{iv}}(0)x^4}{4!} + \cdots \tag{24}$$

This is a general form of Maclaurin's series, so that if a function has a Maclaurin's series, the series can be derived directly from (24). This form should be memorized.†

* It is essential at this point for you to review the functional notation described in Secs. 2-6, 5-12, and 7-6.

† Maclaurin's series can be expressed briefly in such forms as

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where the summation includes all terms of the form

$$\frac{f^{(n)}(0)}{n!} x^n$$

where n involves all positive whole numbers from 1 to ∞, and where $f^{(n)}(0)$ is the value of the nth derivative of $f(x)$ when $x = 0$.

The statement that a function may (under appropriate conditions) be represented in the form (24) is called Maclaurin's theorem; and a function so represented is said to be *expanded according to Maclaurin's theorem.*

It should be emphasized that the value given by the sum of any particular number of terms of Maclaurin's series is not exactly the value of $f(x)$ but an estimate of the limit approached by such a sum as the number of terms is increased.

Notice that from Maclaurin's series we may be able to get the value of a function $f(x)$ for some given value of x—preferably close to zero—*if we already know, or can get*

1. The value of the function when $x = 0$.
2. And the values of all the successive derivatives at the point where $x = 0$.

For many important functions, the above values are easy to get. Some additional examples relating to the Maclaurin's series follow.

Example 1. It is not possible to get an exact formula for

$$\int_0^x \frac{\sin x\, dx}{x}$$

Obtain by a Maclaurin's expansion a method of approximating this integral as closely as desired.

We have already found that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Thus

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

and

$$\int_0^x \frac{\sin x\, dx}{x} = x - \frac{x^3}{3 \times 3!} + \frac{x^5}{5 \times 5!} - \frac{x^7}{7 \times 7!} + \cdots \tag{25}$$

The above integral is but one of many important integrals which cannot be obtained in *closed* form, that is, in the form of a finite number of the *elementary* functions such as power functions, sines, logarithms, exponentials, etc. But whether or not a function is considered elementary depends upon such things as its usefulness at the different mathematical levels and the general availability of tables of its numerical values. Actually, the above integral is called the *sine integral of x* [symbol Si(x)], and its numerical values are published in tables.[3]

Example 2. The voltage in a circuit varied according to $v = 0.2e^{t^2}$ volts over a certain time interval. Obtain a Maclaurin's expansion describing this voltage.

We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Letting $x = t^2$,

$$v = 0.2\left(1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots\right) \qquad \text{volts}$$

QUESTIONS

1. State the difference between a series and the sum of a group of quantities taken at random.
2. Write type forms for (*a*) infinite series in general and (*b*) infinite power series.
3. Give an example of an infinite series of constant terms.
4. Define a convergent series. Give an example.
5. Define a divergent series. Give an example.
6. Is a convergent series necessarily useful for practical applications? If not, what might be a reason?
7. Give a general form for Maclaurin's series.
8. Does every function have a Maclaurin's series? If not, give an example of a characteristic of a function which might not have such a series.
9. The quantity 6! has what value?
10. If a given function is known to be representable by a Maclaurin's series, what information is needed to write this series?
11. Give three further terms for the series representing $\sin x$ as given in (18*a*).
12. Give three further terms for the series representing e^x as stated in (19).
13. Is it possible for $\ln x$ to have a Maclaurin's series? [See comment following Eq. 17).]

PROBLEMS

1. Substituting $x = 1$ in the series for e^x, find e to four significant figures. Use seven terms of the series.

2. The current in an RC circuit followed the equation $i = 0.1e^{-0.001t}$ amperes. Express i as a Maclaurin's series.

3. (*a*) Derive a Maclaurin's series for $\cos x$, getting five terms which do not vanish in the process of derivation. (*b*) Using the series just derived, obtain without tables the value, to five decimal places, of a voltage $v = 22 \cos \omega t$ volts when $\omega t = 0.1$ radian.

4. By means of (18*a*) obtain (*a*) the sine of 0.02 radian and (*b*) the value of a current $i = 1.8 \sin \omega t$ when $\omega t = 0.02$ radian.

5. If $i = 2 \exp t^3$, find without tables the value of i when $t = 0.05$ second.

6. The current supplied to an initially discharged 1-microfarad capacitor varied according to $i = 0.1(1 - \cos t)/t$ amperes. Find the voltage across the capacitor when $t = 0.01$ second.

7. It is desired that the voltage induced in a coil of inductance L over a certain interval should be $v_{ind} = (e^t - 1)/t$ volts. Find the equation of the current which should be sent through the coil to get this result.

8. When a constant current I is supplied to a parallel RC circuit, the voltage across the circuit varies as $v = RI(1 - e^{-t/RC})$ volts. Find the form approached by this voltage wave as R is made larger.

9. It is found that the secondary emf of a transformer, during a certain time interval, is given by $v_2 = 3(\sin t - 1)/t$. Express as a series of the Maclaurin type the varying component of the primary current which must have been flowing.

10. Derive a Maclaurin's series expressing (*a*) sinh x and (*b*) cosh x. (*c*) Through these results, confirm that $e^x = \sinh x + \cosh x$. (*d*) If the height h of a horizontal antenna wire varies with horizontal distance x from the center point according to $h = 100 \cosh 0.01x - 55$ feet, find without the tables the value of h at the point where $x = 10$.

11. Express as a Maclaurin's series the current $i = \sin t^2$ amperes. Then find the amount of charge transmitted by this current from $t = 0$ to $t = 0.03$ second. Express to nearest microcoulomb.

17-8 Existence of Maclaurin's series. In formal courses consideration is given to the conditions which are necessary and sufficient for the representability of a function in a Maclaurin's series. Two conditions, of importance in practical problems, which must be fulfilled before a function can be expressed as a Maclaurin's series are

1. The function must be single-valued.
2. The function and all of its derivatives must exist and be continuous.

These conditions must apply at the point where $x = 0$ and at the point where $f(x)$ is being evaluated by the series, as well as at every point in the interval between. As examples, we cannot expect to find Maclaurin's series to represent functions which become infinite where $x = 0$ or whose graphs have breaks or sharp corners.

As we should expect from the method of arriving at a Maclaurin's series,

➤ A given function has no more than one Maclaurin's series representing it.

Formulas[4] have been developed to show how many terms we must use to reduce the possible error in a Maclaurin's series representation to a given amount.

17-9 Maclaurin's series—graphical treatment. Our comprehension of the significance of a Maclaurin's expansion may be improved by considering the problem graphically. Imagine that a current i varies with time t as shown in Fig. 17-3. But suppose for the moment that we do not possess knowledge of this graph; suppose, in fact, that *we know only the value* $i(0)$ *of* i *when* $t = 0$. Let it be desired to express approximately the value of i when $t = t_1$, where t_1 is close to zero.

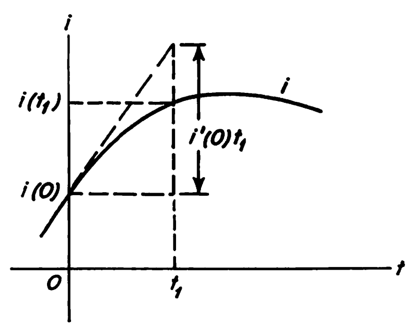


Fig. 17-3

A current cannot change at an infinite rate. Therefore, while the error may be quite large, we may say as a first approximation that

$$i(t_1) = i(0) + \cdots$$

where $i(t_1)$ is the value of i when $t = t_1$, and where the series of dots indicates spaces for the insertion of any *correction* terms which we may be able to discover.

Next, suppose that in some way we are able to find the rate* $i'(0)$ at which i varies when $t = 0$. This may help us to get a closer approximation to $i(t_1)$, for *if i continued to vary at the rate $i'(0)$*, it would increase by an amount $i'(0)t_1$ during the interval from $t = 0$ to $t = t_1$. Our result may be improved if we provide the correction term just found:

$$i(t_1) = i(0) + i'(0)t_1 + \cdots$$

This result, we observe, is the start of a Maclaurin series for $i(t_1)$. Presumably, if we were able to find the values of higher derivatives of i where $t = 0$, we should be able to calculate further terms of the series. These calculations are omitted here.

17-10 Definition of e^z. As we stated near the beginning, the quantities referred to in this book are in most cases limited to real values. But the series expansions just developed allow us to define some functions outside this limitation—functions which we shall find very useful.

We have already seen that e^z can be represented by

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{26}$$

if z is real. In texts on functions of a complex variable it is shown that meaning may also be attached to (26) even when z is imaginary or complex.

At this point, we *define* an exponential e^z, where z can be real, imaginary, or complex, as being *that quantity represented by* (26). Exponentials so defined have most of the familiar properties of real exponentials. For instance, if we multiply two quantities $\exp(jy_1)$ and $\exp(jy_2)$, we get a quantity whose exponent is the *sum* of the two given exponents, so that the product is $\exp[j(y_1+y_2)]$.

Let it be required, as an example, to find the value of e^{jy} (where y itself is real). By the above definition,

$$e^{jy} = 1 + jy + \frac{(jy)^2}{2!} + \frac{(jy)^3}{3!} + \frac{(jy)^4}{4!} + \frac{(jy)^5}{5!} + \cdots$$

Since $j = \sqrt{-1}$, the above series can be written

$$e^{jy} = 1 + jy - \frac{y^2}{2!} - j\frac{y^3}{3!} + \frac{y^4}{4!} + j\frac{y^5}{5!} - \cdots$$

* The *prime* notation for derivatives is customarily used for derivatives *with respect to x*. For the moment, we here use $i'(0)$ to indicate the value of di/dt when $t = 0$.

Separating imaginary from real terms,

$$e^{jy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + j\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

The first expression in parentheses is recognized as the Maclaurin's expansion for cos y (Sec. 17-7, Prob. 3), and the second expression in parentheses is the series for sin y. Thus

$$e^{jy} = \cos y + j \sin y \tag{27}$$

This result should be remembered. It is far-reaching in its applications to electricity.* As an immediate example, consider the circuit of Fig. 17-4*a*. Here a voltage $v = V_{\max} \sin \omega t$ is impressed across a series *RC* circuit. The resulting current has a phase angle θ with respect to v,

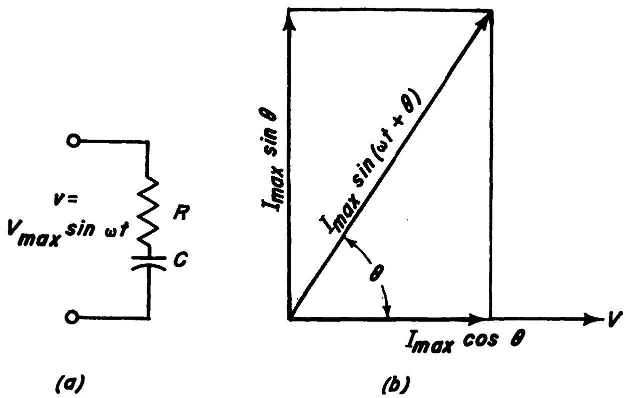


Fig. 17-4

so that the instantaneous current is given by $i = I_{\max} \sin (\omega t + \theta)$, as shown in Fig. 17-4*b*.

According to elementary electrical theory, we can represent this current as being made up of two components. One, the *in-phase* component, is $|\boldsymbol{I}| \cos \theta$, where $|\boldsymbol{I}|$ is the numerical or *scalar* value of current in amperes, without regard to its phase. The other component, a *quadrature* component, is 90° out of phase with the applied voltage and is given by $j|\boldsymbol{I}| \sin \theta$. The vector value† of the current $\boldsymbol{I}$ is then the sum of these two components:

$$\boldsymbol{I} = |\boldsymbol{I}|(\cos \theta + j \sin \theta) \qquad \text{or} \qquad \boldsymbol{I} = |\boldsymbol{I}|e^{j\theta} \tag{28}$$

* The quantity $\cos y + j \sin y$ is often written cis y, and is spoken of as the "cis function of y." Thus $e^{jy} = \text{cis } y$.

† Boldface italic symbols, like $\boldsymbol{I}$, are used in this section to indicate *phasor* quantities, that is, the usual *time* vectors of elementary electric-circuit theory. Symbols like $|\boldsymbol{I}|$ indicate the scalar magnitudes of these quantities. Both kinds of symbols usually represent effective values.

Thus we actually have four ways in which we can represent a sinusoidal function of arbitrary phase and amplitude:

$$\begin{aligned} i &= I_{\max} \sin(\omega t + \theta) \\ I &= |I|(\cos\theta + j\sin\theta) \\ I &= |I|\underline{/\theta} \\ I &= |I|e^{j\theta} \end{aligned} \tag{29}$$

You will note that these are equivalent ways of conveying the same information. The expression which is used in any particular case will depend upon which is the most convenient for the purpose at hand.

Example. A current $I = |I|\underline{/\theta}$ amperes flows in a circuit having an impedance $Z = |Z|\underline{/\phi}$ ohms. Show that the resulting voltage drop across the circuit is $V = |I|\ |Z|\underline{/\theta + \phi}$ volts.

By definition, the impedance of a circuit is $Z = V/I$, so that $V = IZ$. Writing $I = |I|e^{j\theta}$ and $Z = |Z|e^{j\phi}$, in accordance with (29), we get $V = |I|\ |Z|e^{j\theta}e^{j\phi} = |I|\ |Z|e^{j(\theta+\phi)}$ But, again referring to (29), we find this latter expression can be written $V = |I|\ |Z|\underline{/\theta + \phi}$. This illustrates the consistency of our elementary way of multiplying time vectors.

17-11 The Euler identities.

We have seen that

$$e^{jy} = \cos y + j\sin y \tag{27}$$

Consider the effect of expanding a quantity having a *negative* exponent. In a manner similar to that for getting (27), we find

$$e^{-jy} = \cos y - j\sin y \tag{30}$$

If we subtract (30) from (27), we get $e^{jy} - e^{-jy} = j2\sin y$, or

$$\sin y = \frac{e^{jy} - e^{-jy}}{j2} \tag{31}$$

We may also add (30) to (28), getting $e^{jy} + e^{-jy} = 2\cos y$, or

$$\cos y = \frac{e^{jy} + e^{-jy}}{2} \tag{32}$$

In (31) and (32) we have *exponential* representations* of the sine and cosine functions, which are convenient for many uses. Note the similarity to the exponential forms of the hyperbolic functions. You should study the following examples.

Example 1. Many trigonometric formulas can be demonstrated by the use of exponential forms. Show that $\sin^2 y + \cos^2 y = 1$.

* Formulas (31) and (32) are called the Euler identities, after Leonhard Euler, famous Swiss mathematician (1707–1783).

Squaring (31), we get $\sin^2 y = -(e^{j2y} - 2 + e^{-j2y})/4$. Squaring (32), we have $\cos^2 y = (e^{j2y} + 2 + e^{-j2y})/4$. Adding these results, we get the required identity.

Example 2. In other courses it is shown that many of the functions of a complex variable can be differentiated or integrated according to the formulas used for real variables.[5] Carry out the following integration, which was encountered in a study of electric transients:

$$\int_0^\infty \sin \beta t \, e^{-st} \, dt$$

(Here t itself is real, s is complex.)

Making the substitution (31), we get

$$\int_0^\infty \frac{e^{j\beta t} - e^{-j\beta}}{j2} e^{-st} \, dt = \frac{1}{j2} \int_0^\infty [e^{-(s-j\beta)t} - e^{-(s+j\beta)t}] \, dt$$

We now integrate this quantity term by term, getting

$$-\frac{1}{j2}\left[\frac{e^{-(s-j\beta)t}}{s - j\beta} - \frac{e^{-(s+j\beta)t}}{s + j\beta}\right]_0^\infty \qquad \text{or} \qquad \frac{1}{j2}\left(\frac{1}{s - j\beta} - \frac{1}{s + j\beta}\right)$$

This simplifies to

$$\frac{\beta}{s^2 + \beta^2}$$

QUESTIONS

1. What meaning is attached to the expression e^{jy}?
2. Give four different notations expressing a wave of sinusoidal form having a phase angle θ with respect to some reference wave.
3. What consideration decides which of the four forms mentioned in question 2 shall be used in any particular case?
4. With reference to the introductory sentences of this chapter, state two general uses for infinite series. Give examples of each of these uses as they have been developed thus far.
5. Express $\sin x$ as an exponential form [Formula (31)]. Similarly for $\cos x$ [Formula (32)].

PROBLEMS

1. Demonstrate, through (29), our elementary rule that

$$\frac{A\underline{/\theta}}{B\underline{/\phi}} = \frac{A}{B}\underline{/\theta - \phi}$$

2. The current in a circuit was $i = 65 \cosh (j\omega t)$. (*a*) Find a Maclaurin's series for $\cosh (j\omega t)$. (*b*) Compare with the Maclaurin's series for $\cos \omega t$ (Sec. 17-7, Prob. 3). Express the above current in trigonometric form.
3. In studying an oscillatory current we find it necessary to know that $\sinh j\theta = j \sin \theta$. Prove this identity, using (31).
4. The input impedance of a certain antenna is

$$Z = \left(\frac{L}{C}\right)^{1/2} \frac{\cosh [j\omega(LC)^{1/2}]}{\sinh [j\omega(LC)^{1/2}]}$$

where L and C are the inductance and capacitance per unit length of the antenna. Show that this is equivalent to $Z = -j(L/C)^{1/2} \cot \omega(LC)^{1/2}$. (HINT: Use the results of Probs. 2 and 3.)

5. The current in a circuit is $i = 20(\sin \omega t \cos \theta + \cos \omega t \sin \theta)$. (*a*) Using (31) and (32), find an identity expressing this form as a single trigonometric function involving both ωt and θ. (*b*) Use the resulting identity to express the above current.

6. The voltage applied to a circuit is $v = V_{max} \sin \omega t$. The resulting current is $i = I_{max} \sin (\omega t + \theta)$. Show through (31) that the instantaneous power is $p = I_{eff}V_{eff} [\cos \theta - \cos (2\omega t + \theta)]$. (NOTE: This result shows that the instantaneous power varies as a sinusoidal wave of twice the frequency of the applied voltage superimposed upon a steady value $I_{eff}V_{eff} \cos \theta$, which is the average power dissipated in the circuit.)

7. In solving a problem in electric transients it was required to evaluate

$$\int_0^\infty \cos \beta t \, e^{-st} \, dt$$

where t is real and s is complex. Carry out this integration.

17-12 Brief exponential form of sine and cosine functions. From the identity

$$e^{jy} = \cos y + j \sin y$$

we get

$$\begin{aligned} \sin y &= \text{imaginary part of } e^{jy} \\ \cos y &= \text{real part of } e^{jy} \end{aligned}$$

These expressions can be written in shorter form, with no change in meaning, by use of the symbols Im and Re:

$$\begin{aligned} \sin y &= \text{Im } e^{jy} \\ \cos y &= \text{Re } e^{jy} \end{aligned} \tag{33}$$

Here Im and Re are *operators* which indicate that only the imaginary part or the real part, respectively, is to be taken of the quantity to which they are applied.*

Through (33) we are led to even simpler exponential forms for $\sin y$ and $\cos y$ than those of (31) and (32). These new and shorter forms are sufficient for a large majority of practical cases. Their usefulness depends upon these facts:

1. The real part of the sum of two quantities is equal to the sum of the real parts of these quantities, and correspondingly for the imaginary parts. Thus $(6 + j5) + (2 - j3) = 8 + j2$.

2. The real part of the derivative (or integral) of a complex quantity is the derivative (or integral) of the real part of the quantity, and correspondingly for the imaginary parts. Thus, where x and y are themselves real,

$$\text{Im} \frac{d}{dt}(x + jy) = \frac{dy}{dt}$$

* Other symbols for Im are I and $\mathfrak{I}$. Other symbols for Re are R and $\mathfrak{R}$.

It must be noted, however, that the above "commutative" process does not apply to multiplication or division. Thus,

$$(2 + j3)(4 + j2) = 2 + j16$$

so that we do *not* get the real part of the product, for instance, simply by multiplying 2×4.

Now we are ready to *apply* the brief complex form for a sine or cosine function. As a very simple example, let us derive the familiar relation

$$\int \sin \theta \, d\theta = -\cos \theta + C$$

Writing Im $e^{j\theta}$ instead of $\sin \theta$,

$$\begin{aligned}\int \sin \theta \, d\theta &= \int \mathrm{Im}\,(e^{j\theta})\, d\theta = \mathrm{Im} \int e^{j\theta}\, d\theta = \mathrm{Im}\left(\frac{1}{j}\, e^{j\theta}\right) + C \\ &= \mathrm{Im}\,(-je^{j\theta}) + C\end{aligned}$$

Now instead of $e^{j\theta}$ we may write $\cos \theta + j \sin \theta$. The above result becomes

$$\int \sin \theta \, d\theta = \mathrm{Im}\,[-j(\cos \theta + j \sin \theta)] + C = \mathrm{Im}\,(-j \cos \theta + \sin \theta) + C$$

The operator Im tells us to select from this result the quantity $-\cos \theta$, since this is the imaginary part of the quantity in the parentheses:

$$\int \sin \theta \, d\theta = -\cos \theta + C$$

The exponential form for the sine or cosine is of doubtful value in such a simple case as this. Its usefulness seems to grow as the difficulty of the problem increases. After you have gained some familiarity with this form, you can in fact drop the Im or Re notation throughout, for you will know whether to take the real or the imaginary part of the result simply by noting whether $e^{j\theta}$ was used originally to represent the sine or the cosine function.

Example. The current in a circuit was $i = e^{at} \cos bt$. Integrate this form with respect to t.

Writing e^{jbt} for $\cos bt$,

$$\int e^{at} \cos bt \, dt = \int e^{at} e^{jbt}\, dt = \int e^{(a+jb)t}\, dt = \frac{1}{a + jb}\, e^{(a+jb)t} + C$$

Rationalizing the denominator of the fraction, we have

$$\int e^{at}\, e^{jbt}\, dt = \frac{e^{at}}{a^2 + b^2}\,(a - jb)e^{jbt} + C$$

Making the substitution $e^{jbt} = \cos bt + j \sin bt$, we get

$$\int e^{at} e^{jbt}\, dt = \frac{e^{at}}{a^2 + b^2}(a - jb)(\cos bt + j \sin bt) + C$$

$$= \frac{e^{at}}{a^2 + b^2}[a \cos bt + b \sin bt + j(a \sin bt - b \cos bt)] + C$$

Now, since e^{jbt} originally represented $\cos bt$, we select the real part of the above result:

$$\int e^{at} \cos bt\, dt = \frac{e^{at}}{a^2 + b^2}(a \cos bt + b \sin bt) + C$$

Had we taken the imaginary part of the answer, we would have had a solution for $\int e^{at} \sin bt\, dt$. In the absence of tables, the alternative to the above procedure would have been a double integration by parts as described in Chap. 15. The present method is by comparison quite easy.

Caution must be used in applying this brief complex notation to problems where several sine or cosine quantities, or other quantities having imaginary components, appear together.* But the simple form e^{jy} can be used in a large number of cases to represent $\sin y$ or $\cos y$, and because of the ease of differentiating and integrating the exponential form, the labor of many calculations is reduced. This notation is particularly useful in solving linear ac circuits in the steady state, that is, circuits whose inductance, resistance, and capacitance do not vary and whose currents and voltages are steady sinusoidal waves.

QUESTIONS

1. What is the meaning of the operator Im? Of the operator Re?
2. Express $\sin x$ as a single exponential, indicating by the proper operator whether a real or an imaginary part of the exponential is intended. Similarly for $\cos x$.
3. For what class of electrical problems are exponential forms for sinusoidal waves especially useful?

PROBLEMS

1. The current delivered to an initially discharged 20-microfarad capacitor was $i = 0.05e^{-400t} \sin 100\pi t$, where t was in seconds. Find the voltage across the capacitor after 10 milliseconds. Use (33).

2. In solving a series RL circuit the following quantity had to be evaluated:

$$\frac{V}{L} e^{-Rt/L} \int e^{Rt/L} \sin \omega t\, dt$$

Find the result by (33).

* This warning is necessary because, as previously pointed out, the real parts of products (or quotients) are not obtained directly from products (or quotients) of real parts. It is only in advanced problems that such questions often arise. In such cases, the more complete exponential forms of (31) and (32) should be considered.

3. A current $i = 1.2e^{-100t} \sin 250\pi t$ was sent through a 2-henry inductor (t was in seconds). Neglecting resistance, find the voltage across the inductor when $t = 2$ milliseconds. Use (33).

4. The current in a circuit is $I_{\max} \sin \omega t$. The resulting voltage drop across the circuit is $V_{\max} \sin (\omega t + \theta)$. (*a*) Express these quantities in complex form in accordance with (33). (*b*) Find the ratio of voltage to current, showing that the functions of time disappear in this ratio, leaving a complex exponential function of θ, independent of time. (*c*) If we take this ratio to be the impedance of the circuit, what quantity would properly represent the phase angle of the impedance?

5. Over a certain interval the voltage induced in a transformer secondary was $v_2 = \cosh at \sin \omega t$. Find the form of the current which flows in the primary to produce this result.

6. The current in a circuit was $i = 0.21 \sin \omega t \sinh \omega t$. Find the charge transmitted in the interval from $\omega t = 0$ to $\omega t = \pi$.

REFERENCES

1. H. M. Bacon: "Differential and Integral Calculus," 2d ed., chap. 19, McGraw-Hill Book Company, Inc., New York, 1955.
2. E. J. Townsend and G. A. Goodenough: "Essentials of Calculus," 2d ed., pp. 307–308, Henry Holt and Company, New York, 1910.
3. A. N. Lowan (Technical Director, Works Projects Administration for the City of New York): "Tables of Sine, Cosine and Exponential Integrals," vols. I and II, sponsored by National Bureau of Standards, Washington, D.C.
4. F. L. Griffin: "Mathematical Analysis: Higher Course," pp. 249–256, Houghton Mifflin Company, Boston, 1927.
5. J. Pierpont: "Functions of a Complex Variable," chap. 6, Ginn & Company, Boston, 1914.

18

Taylor's Series

Here we consider a power series which is more general than the Maclaurin's series just studied. This is Taylor's series, one of the most useful topics of calculus.

18-1 Taylor's series. Let it be desired to express a function $f(x)$ as an infinite series of powers of the quantity $x - a$, where a is some constant. If this series exists, its form will be

$$f(x) = A + B(x - a) + C(x - a)^2 + D(x - a)^3 + \cdots \tag{1}$$

where A, B, C, etc., are coefficients which are to be determined. Let us differentiate (1) several times:

$$f'(x) = B + 2C(x - a) + 3D(x - a)^2 + \cdots \tag{2}$$

$$f''(x) = 2C + 6D(x - a) + \cdots \tag{3}$$

$$f'''(x) = 6D + \cdots \tag{4}$$

Assume that (1) to (4) apply when $x = a$. Actually letting $x = a$ for the moment and solving for A, B, etc.,

$$f(a) = A \qquad A = f(a) \tag{1a}$$

$$f'(a) = B \qquad B = f'(a) \tag{2a}$$

$$f''(a) = 2C \qquad C = \frac{f''(a)}{2!} \tag{3a}$$

$$f'''(a) = 6D \qquad D = \frac{f'''(a)}{3!} \qquad \text{etc.} \tag{4a}$$

Substituting these values of A, B, etc., into (1),

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \frac{f'''(a)(x - a)^3}{3!} + \cdots \tag{5}$$

This is Taylor's series, and a function $f(x)$ so expressed is said to be expanded according to Taylor's theorem.* This expansion was published in 1715 by Dr. Brook Taylor of England. From both the practical and the theoretical viewpoints, Taylor's theorem is of fundamental importance in calculus.

Through Taylor's series, we may be able to get the value of a function $f(x)$ for some given value of x, if we have

1. The value $f(a)$ of the function for some nearby value a of the independent variable.
2. And the values $f'(a)$, $f''(a)$, etc., of all the successive derivatives of $f(x)$ at the point where $x = a$.

Example 1. Express $\sin x$ as a Taylor's series.
We write

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \qquad \text{etc.} \end{aligned}$$

These values make (5) read as follows:

$$\sin x = \sin a + \cos a(x - a) - \frac{\sin a(x - a)^2}{2!} - \frac{\cos a(x - a)^3}{3!} + \cdots \tag{6}$$

We see that, in the special case where $a = 0$, the Taylor's series (5) becomes identical to the Maclaurin's series [Eq. (24) of Sec. 17-7]. Advantages which may be possessed by the Taylor's series include these:

* Another expression is

$$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)(x - a)^n}{n!}$$

1. It may exist for functions having no Maclaurin's series.

2. Even when the Maclaurin's series exists, the Taylor's series may converge more rapidly for suitable values of a than does the Maclaurin's series for the same function. The key to this advantage is seen in the powers of $x - a$ which appear in (5). If x lies near to a, the higher powers of $x - a$ will not grow rapidly, so that the later terms in the series may decrease speedily because of the very large denominators. (In case x lies near to zero, the advantage is likely to lie with the Maclaurin's series if it exists.)

Example 2. Find sin 0.7859 radian (*a*) by a Taylor's series and (*b*) by a Maclaurin's series.

For the first solution, we choose to expand sin 0.7859 as a Taylor's series in powers of $0.7859 - \pi/4$. The choice $a = \pi/4$ was made because

1. The successive derivatives of $\sin x$ are $\cos x$, $-\sin x$, $-\cos x$, etc., and we know not only the value of $\sin x$ but the values of these derivatives where $x = \pi/4$ radians ($= 45°$). These values are, of course, $\sqrt{2}/2$ ($= 0.707107$) preceded by the proper signs.

2. The value of $\pi/4$ ($= 0.785398$) is close to the value of x, that is, 0.7859, for which we desire the sine function. This makes $x - a = 0.7859 - 0.785398 = 0.000502$, and this is desirably small.

Our Taylor's series is then

$$\sin 0.7859 = 0.707107 + (0.707107)(0.000502) - \frac{(0.707107)(0.000502)^2}{2!} + \cdots$$

It is seen that in this case only the first two terms can affect the fifth decimal place in the result. Taking only these two terms, we get

$$\sin 0.7859 \text{ radian} = 0.70746$$

The corresponding Maclaurin's series is

$$\sin 0.7859 = 0.7859 - \frac{0.7859^3}{3!} + \frac{0.7859^5}{5!} - \frac{0.7859^7}{7!} + \cdots$$

If we use all four of the terms shown, we get the same result as by two terms of the Taylor's series, but the calculations are longer.

Example 3. A voltage of angular frequency ω is applied to a series RLC circuit. The net reactance is then $X = \omega L - 1/\omega C$. Find a simple approximate expression for X at frequencies near the resonant frequency ω_0.

At resonance

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{7}$$

Since X is a function of ω, we may represent it by $f(\omega)$:

$$f(\omega) = \omega L - \frac{1}{\omega C} \tag{8}$$

Expanding (8) in a Taylor's series,

$$f(\omega) = f(\omega_0) + f'(\omega_0)(\omega - \omega_0) + \frac{f''(\omega_0)(\omega - \omega_0)^2}{2!} + \cdots \tag{9}$$

For cases in which ω is near to ω_0, terms after the second term in the right member may be neglected. Further, the net reactance $f(\omega_0)$ at the resonant frequency is zero. Then (9) becomes

$$f(\omega) \approx f'(\omega_0)(\omega - \omega_0) \tag{10}$$

Differentiating (8), we get $f'(\omega) = L + 1/\omega^2 C$, so that

$$f'(\omega_0) = L + \frac{1}{\omega_0{}^2 C} \tag{11}$$

From (7), $1/\omega_0{}^2C = L$, so that (11) becomes $f'(\omega_0) = 2L$. This makes (10)

$$X = f(\omega) \approx 2L(\omega - \omega_0)$$

18-2 Existence of Taylor's series. Formal treatment of the conditions which are necessary and sufficient for the expressibility of a function in a Taylor's series is left to other texts. We shall only say that for the existence of a Taylor's series corresponding to $f(x)$, this function and all of its derivatives must exist (a) where $x = a$ and (b) at the point x where we are evaluating $f(x)$, as well as at all points between. These requirements are sufficient for the existence of the series in nearly all cases of practical interest.

18-3 Taylor's series—graphical treatment. Consider a current i which varies as shown in Fig. 18-1. Suppose now that we do not have a

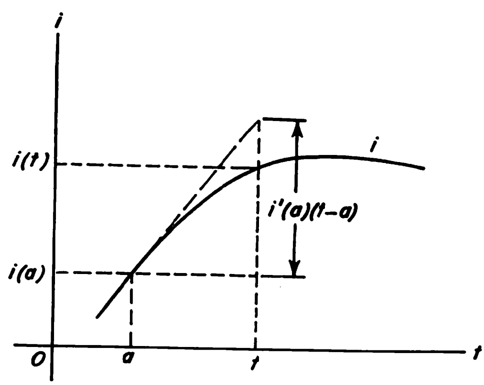


Fig. 18-1

knowledge of this graph but that we do know the value $i(a)$ of the current when $t = a$. Let it be desired to express approximately the value $i(t)$ taken on by i when t has some value other than a.

Since a current cannot change at an infinite rate, $i(a)$ might be a rather good first approximation of $i(t)$ if t is not greatly different from a. We write

$$i(t) = i(a) + \cdots$$

where the series of dots provides for the insertion of any correction terms which we may be able to find.

Next, suppose that we are able to find the rate $i'(a)$ at which i varies when $t = a$. If i continued to vary at the rate $i'(a)$, it would change by an amount $i'(a)(t - a)$ while the time changes from a to t. Inserting this correction

$$i(t) = i(a) + i'(a)(t - a) + \cdots$$

This is the beginning of a Taylor's series for $i(t)$. Further terms might be derived by a more complicated reasoning. Observe that for the case $a = 0$ this presentation is directly comparable with that of Sec. 17-9 for the Maclaurin's series.

QUESTIONS

1. Give a general form for Taylor's series.
2. What are two conditions which must apply in order that a function may be representable in a Taylor's series?
3. If a given function is known to be representable in a Taylor's series, what information is needed to write this series?
4. Under what conditions does the Taylor's series take the particular form called Maclaurin's series?
5. State for which of the following functions it may be possible to write a Taylor's series, and, in case the Taylor's series does not exist, state why it does not exist (relative to the answer to question 2):

(*a*) A function which takes the value $-\infty$ when $x = a$.

(*b*) A function which is equal to zero when $x = a$.

(*c*) A function which takes an abrupt vertical jump at some point between a and x.

(*d*) A function which takes an abrupt vertical drop where $x = a$.

(*e*) A function which has a sharp corner at the junction of two smooth curves between a and x.

(*f*) A function which is equal to zero at some point between x and a.

(*g*) A function whose graph is flat for a short space between x and a.

(*h*) The function $\tan x$ if a is less than $\pi/2$ and x is greater than $\pi/2$.

(*i*) A sawtooth wave, theoretically perfect in form, if the "retrace" begins between a and t.

PROBLEMS

1. (*a*) Express sin 0.5 radian as a Taylor's series in powers of $0.5 - \pi/6$. (*b*) Given that $\sin \pi/6 = 0.5$, that $\cos \pi/6 = \sqrt{3}/2$ $(= 0.866025)$, and that $\pi/6 = 0.523599$, find sin 0.5 radian to four significant figures. Use three terms of the series.

2. Express $\cos x$ as a Taylor's series in powers of $x - a$.

3. (*a*) Express $\ln x$ as a Taylor's series in powers of $x - a$. (*b*) Rewrite this series letting $a = 1$, remembering that $\ln 1 = 0$. Compare with Eq. (11), Sec. 17-4, where $x + 1$ corresponds to x in the present problem. (*c*) Use the series of part (*b*) to obtain ln 1.1.

4. (*a*) Write e^x as a Taylor's series in terms of $x - a$. (*b*) Rewrite this series letting $a = 2$. (*c*) If we are given that $e^2 = 7.38906$, find $e^{2.08}$.

5. Write a Taylor expansion of $\sinh x$ in terms of $x - a$.

6. A capacitance C is charged to V volts, then discharged through a resistance R. Show that for values of time t close to RC seconds, the voltage across the capacitor is approximately $v = V(2RC - t)/RCe$ volts.

7. A signal has power P watts. Its level L decibels with respect to a fixed reference power P_r is defined as $L = 10 \log_{10} (P/P_r) = 10M(\ln P - \ln P_r)$. Show that for values of P close to P_r, $L \approx 10M(P - P_r)/P_r$.

18-4 Indeterminate forms. Consider the quotient

$$y = \frac{f(x)}{F(x)} \tag{12}$$

Suppose that, when x takes the value a, both $f(x)$ and $F(x)$ become equal to zero:

$$f(a) = F(a) = 0 \tag{13}$$

In this case, we say that the function y takes the *indeterminate form* $0/0$ when $x = a$. This means, specifically, that (12) *does not define* a value of the function y for that point on the graph of y where $x = a$.

Mathematicians reserve to themselves, however, the privilege of *assigning* to a function any value(s) whatsoever that may suit their purposes, for any point (or set of points) on the graph of the function. And in many cases, a function like y in (12) above may approach some *limit* as x approaches a. In such cases, it is often useful to *assign* to y the value of this limit for the case where $x = a$.

In order to evaluate this limit, if it exists, let us suppose that both $f(x)$ and $F(x)$ can be expanded in Taylor's series involving powers of $x - a$:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \cdots \tag{14}$$

$$F(x) = F(a) + F'(a)(x - a) + \frac{F''(a)(x - a)^2}{2!} + \cdots \tag{15}$$

By (13), the first terms in the right members of (14) and (15) vanish. If we divide (14) by (15), we get, after dividing numerator and denominator by $x - a$,

$$y = \frac{f(x)}{F(x)} = \frac{f'(a) + f''(a)(x - a)/2! + \cdots}{F'(a) + F''(a)(x - a)/2! + \cdots} \tag{16}$$

If in (16) we let x approach a, we get

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \frac{f'(a)}{F'(a)} \tag{17}$$

unless the right member of (17) is itself indeterminate.

If the right member of (17) also takes the indeterminate form $0/0$, we apply the same process to that member, getting

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x)}{F'(x)} = \frac{f''(a)}{F''(a)} \tag{18}$$

If this again gives us an indeterminate form $0/0$, we continue, using successive derivatives until a result is found which is not indeterminate [if the original quotient $f(x)/F(x)$ indeed approaches a limit as x approaches a].

Example 1. What value would probably be assigned to $y = (\sin x)/x$ for the case where $x = 0$?

When $x = 0$, this function takes the meaningless form $0/0$. But as x approaches 0, the function y in this case happens to approach a limit, and we may usefully assign to y the value of that limit when $x = 0$. To evaluate this limit, we find

$$f'(0) = \frac{d}{dx} \sin x \text{ (when } x = 0) = \cos 0 = 1$$

$$F'(0) = \frac{d}{dx} x \text{ (when } x = 0) = 1$$

Thus $$\lim_{x \to 0} y = \lim_{x \to 0} \frac{\sin x}{x} = 1/1 = 1$$

so that we are likely to assign to y the value 1 when $x = 0$.

Consider now a different case. Suppose that

$$y = \frac{f(x)}{F(x)} \tag{19}$$

and that both $f(x)$ and $F(x)$ become *infinite* when $x = a$. In higher courses it is shown that in this case also

$$\lim_{x \to a} y = \frac{f'(a)}{F'(a)} \tag{20}$$

unless the right member of (20) is itself indeterminate.

If the right member of (20) takes one of the indeterminate forms $0/0$ or ∞/∞, we again apply the same process to that member:

$$\lim_{x\to a}\frac{f(x)}{F(x)} = \lim_{x\to a}\frac{f'(x)}{F'(x)} = \frac{f''(a)}{F''(a)} \tag{21}$$

and so on, if necessary, using successive derivatives until a result is found which is not indeterminate [if the original quotient $f(x)/F(x)$ indeed approaches a limit as x approaches a].

Example 2. If $y = (\ln x)/x$, evaluate $\lim\limits_{x\to\infty} y$.

The given function y takes the form ∞/∞ when x becomes infinite. But y happens to approach a limit as x increases without bound. To evaluate this limit, we write

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\frac{1/x}{1} = 0$$

The procedures outlined above may be condensed into

L'Hôpital's Rule: If $f(x)/F(x)$ takes one of the indeterminate forms $0/0$ or ∞/∞ when $x = a$ (a finite or infinite), then

$$\lim_{x\to a}\frac{f(x)}{F(x)}$$

if this limit exists, is equal to the *first* of the following expressions which is not indeterminate:

$$\lim_{x\to a}\frac{f'(x)}{F'(x)},\ \lim_{x\to a}\frac{f''(x)}{F''(x)},\ \lim_{x\to a}\frac{f'''(x)}{F'''(x)},\ \cdots$$

If none of these latter limits exists, then neither does $\lim\limits_{x\to a}[f(x)/F(x)]$.

Note these comments:

1. We must differentiate the numerator *and* the denominator in the given function, not the function as a quotient.

2. We must apply this rule only to functions which take one of the indeterminate forms $0/0$ or ∞/∞.*

3. In cases where y takes the form ∞/∞ and where a is finite, we must eventually convert to the form $0/0$ by some simplification or form change.

4. The finding of a limit corresponding to an indeterminate form is called *evaluating* that form. However, we are unable to solve for any actual value of a function y which takes the form $0/0$ or ∞/∞ when $x = a$, for such a function, of itself, has no value whatsoever when $x = a$. But we may be able to evaluate by L'Hôpital's rule the *limit* approached by such a function as x approaches a; and it may be desirable to assign to such a function the value of this limit for the case where $x = a$.

* For the forms $0 \times \infty$, $\infty - \infty$, ∞^0, and 1^∞, see ref. 1.

PROBLEMS

Evaluate the following limits.

1. $\lim\limits_{x\to 2} \dfrac{x^2 - 4}{x - 2}$
2. $\lim\limits_{x\to 0} \dfrac{\sin 5x}{x}$
3. $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{x}$
4. $\lim\limits_{x\to 0} \dfrac{1 - e^x}{x}$
5. $\lim\limits_{x\to \pi} \dfrac{\cos x + 1}{x \sin x}$
6. $\lim\limits_{x\to \infty} \dfrac{e^x + 1}{e^x - 1}$
7. $\lim\limits_{x\to \infty} \dfrac{x^2}{e^x}$
8. $\lim\limits_{x\to 0} \dfrac{1 - \sin x}{x^2}$
9. $\lim\limits_{x\to 0} \dfrac{\ln x}{\cot x}$
10. $\lim\limits_{x\to 1} \dfrac{x \ln x + x - x^2}{(x - 1)^2}$

18-5 Taylor's series—second form. In the series (5) let $x - a$ be represented by h. This makes $x = a + h$, and (5) becomes

$$f(a + h) = f(a) + f'(a)h + \frac{f''(a)}{2!} h^2 + \frac{f'''(a)}{3!} h^3 + \cdots \tag{22}$$

Form (22) is useful in displaying a function of the sum of a variable h and a constant a.

Example 1. Get a Taylor's series for e^{1+h} in powers of h. From this series, obtain the value of $e^{1.1}$.

In (22) let $a = 1$ and let $f(a + h) = e^{1+h}$:

$$e^{1+h} = e + eh + \frac{eh^2}{2!} + \frac{eh^3}{3!} + \cdots = e\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right)$$

This is the desired series. When $h = 0.1$, the series yields

$$e^{1.1} = e\left[1 + 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} + \cdots\right] = 1.1052e = 3.0042$$

Example 2. In an audio-amplifier stage the plate current i_b is a function of the instantaneous grid voltage. This grid voltage is the sum of the signal voltage v_g and the fixed bias voltage V_{c0}. Thus $i_b = f(v_g + V_{c0})$. In a certain amplifier the operation is nonlinear, so that

$$i_b = f(v_g + V_{c0}) = I_{b0} + Av_g + Bv_g{}^2 + Cv_g{}^3 + \cdots \tag{23}$$

Here I_{b0} is the quiescent plate current, and A, B, etc., are constants. Let the input signal consist of the sum of two sinusoidal waves:

$$v_g = V_1 \sin \omega_1 t + V_2 \sin \omega_2 t \tag{24}$$

Find the approximate form of the output wave.

Substituting (24) into (23), we get

$$i_b = I_{b0} + AV_1 \sin \omega_1 t + AV_2 \sin \omega_2 t + BV_1{}^2 \sin^2 \omega_1 t + BV_2{}^2 \sin^2 \omega_2 t + 2BV_1V_2 \sin \omega_1 t \sin \omega_2 t + \cdots \tag{25}$$

But a trigonometric identity states that $\sin^2 x = (1 - \cos 2x)/2$. This puts the fourth and fifth terms of the series (25) into the forms

$$\frac{BV_1^2}{2} - \frac{BV_1^2}{2}\cos 2\omega_1 t \qquad \text{and} \qquad \frac{BV_2^2}{2} - \frac{BV_2^2}{2}\cos 2\omega_2 t \tag{26}$$

respectively. Another identity states that $\sin x \sin y = [\cos (x - y) - \cos (x + y)]/2$. This puts the sixth term of the series (25) into the form

$$BV_1V_2 \cos (\omega_1 - \omega_2)t - BV_1V_2 \cos (\omega_1 + \omega_2)t \tag{27}$$

Making the substitutions (26) and (27), we make (25)

$$\begin{aligned} i_b = I_{b0} &+ \frac{BV_1^2}{2} + \frac{BV_2^2}{2} + AV_1 \sin \omega_1 t + AV_2 \sin \omega_2 t - \frac{BV_1^2}{2}\cos 2\omega_1 t \\ &- \frac{BV_2^2}{2}\cos 2\omega_2 t + BV_1V_2 \cos (\omega_1 - \omega_2)t - BV_1V_2 \cos (\omega_1 + \omega_2)t + \cdots \end{aligned} \tag{28}$$

The second and third terms of (28) indicate the amount of change in direct plate current caused by application of the signal to the grid. The fourth and fifth terms of (28) signify the amplified input signal waves. The sixth and seventh terms are second-harmonic distortion terms resulting from the nonlinear operation of the tube upon the input waves. The eighth and ninth terms represent frequencies which are, respectively, the difference and the sum of the input signal frequencies. In audio amplifiers, these waves (represented by the eighth and ninth terms) represent a particularly vicious form of distortion called *intermodulation distortion.* Since they are not closely related musically to the input frequencies, they may clash violently with those frequencies from the standpoint of the listener. In other circuits, the intermodulation terms may be useful.

Series (28) shows the output current resulting when only the first three terms of the plate-current series (23) are taken into account. If the operation of the tube requires further terms of (23) for its representation, or if an input signal including three or more sine waves is considered, then additional dc, harmonic, and intermodulation terms can be calculated. In particular, the second-degree term Bv_g^2 in (23) gives rise to second-harmonic distortion; the third-degree term Cv_g^3 is responsible for third-harmonic distortion, etc.

18-6 Taylor's series for functions of two variables. It may be shown[2,3] that a function of two or more separate variables can be expanded in a Taylor's series.

Let it be desired to express as a series the value of a function $f(x,y)$. The result turns out as follows, where x_0 and y_0 are known fixed values of x and y, and where the partial derivatives are to be evaluated at (x_0,y_0):*

* It may be desirable to note that, within the brackets, the coefficients of the various partial derivatives are those given by the *binomial theorem,* presented in texts on college algebra. For a convenient *operational* notation, see ref. 3.

$$
\begin{aligned}
f(x,y) = f(x_0,y_0) &+ \left[\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)\right] \\
&+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(y - y_0)^2\right] \\
&+ \frac{1}{3!}\left[\frac{\partial^3 f}{\partial x^3}(x - x_0)^3 + 3\frac{\partial^3 f}{\partial x^2\,\partial y}(x - x_0)^2(y - y_0)\right. \\
&\left. + 3\frac{\partial^3 f}{\partial x\,\partial y^2}(x - x_0)(y - y_0)^2 + \frac{\partial^3 f}{\partial y^3}(y - y_0)^3\right] + \cdots \qquad (29)
\end{aligned}
$$

Example 1. Express as a Taylor's series $e^x \cos y$, letting

$$x_0 = 0 \quad \text{and} \quad y_0 = 0 \qquad (30)$$

Include terms as far as the second set of square brackets in (29).

The various partial derivatives and the values they assume when $x = y = 0$ are calculated below:

$$
\begin{aligned}
&\frac{\partial f}{\partial x} = e^x \cos y\ (= 1) \qquad \frac{\partial f}{\partial y} = -e^x \sin y\ (= 0) \\
&\frac{\partial^2 f}{\partial x^2} = e^x \cos y\ (= 1) \qquad \frac{\partial^2 f}{\partial y^2} = -e^x \cos y\ (= -1) \qquad (31) \\
&\frac{\partial^2 f}{\partial x\,\partial y} = -e^x \sin y \qquad (= 0)
\end{aligned}
$$

Substituting (30) and (31) in (29),

$$e^x \cos y = 1 + x + \frac{x^2}{2!} - \frac{y^2}{2!} + \cdots$$

A second form of (29) may be derived, giving the value $f(x_0 + h, y_0 + k)$ taken on by $f(x,y)$ when

1. x has a value $x_0 + h$ near to a given fixed value x_0.
2. And y has a value $y_0 + k$ near to a given fixed value y_0. This form is

$$
\begin{aligned}
&f(x_0 + h, y_0 + k) \\
&\quad = f(x_0,y_0) + \left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k\right) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}h^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}hk + \frac{\partial^2 f}{\partial y^2}k^2\right) \\
&\quad + \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}h^3 + 3\frac{\partial^3 f}{\partial x\,\partial y^2}hk^2 + 3\frac{\partial^3 f}{\partial x^2\,\partial y}h^2k + \frac{\partial^3 f}{\partial y^3}k^3\right) + \cdots \qquad (32)
\end{aligned}
$$

where again the partial derivatives are evaluated at (x_0,y_0).

Example 2. The plate current i_b of an electron tube is a function of $V_{b0} + v_p$ and of $V_{c0} + v_g$, where V_{b0} and V_{c0} are the quiescent plate and grid voltages, respectively, and where v_p and v_g are the varying (or signal) components of plate and grid voltage, respectively. Express i_b as a Taylor's series of form (32).

We let

$$\begin{aligned} x &= V_{b0} + v_p \ (= v_b) \qquad & x_0 &= V_{b0} \qquad & h &= v_p \\ y &= V_{c0} + v_g \ (= v_c) \qquad & y_0 &= V_{c0} \qquad & k &= v_g \end{aligned}$$

These put (32) in the form

$$\begin{aligned} i_b &= f(V_{b0} + v_p,\ V_{c0} + v_g) \\ &= f(V_{b0}, V_{c0}) + \left(v_p \frac{\partial i_b}{\partial v_b} + v_g \frac{\partial i_b}{\partial v_c} \right) + \frac{1}{2} \left(v_p{}^2 \frac{\partial^2 i_b}{\partial v_b{}^2} + 2 v_p v_g \frac{\partial^2 i_b}{\partial v_b\, \partial v_c} + v_g{}^2 \frac{\partial^2 i_b}{\partial v_c{}^2} \right) \\ &\quad + \frac{1}{6} \left(v_p{}^3 \frac{\partial^3 i_b}{\partial v_b{}^3} + 3 v_p{}^2 v_g \frac{\partial^3 i_b}{\partial v_b{}^2\, \partial v_c} + 3 v_p v_g{}^2 \frac{\partial^3 i_b}{\partial v_b\, \partial v_c{}^2} + v_g{}^3 \frac{\partial^3 i_b}{\partial v_c{}^3} \right) + \cdots \end{aligned} \tag{33}$$

We now evaluate quantities in this series. It is seen that

$$f(V_{b0}, V_{c0}) = I_{b0} \tag{34}$$

which is the quiescent plate current. And

$$\frac{\partial i_b}{\partial v_b} = \frac{1}{r_p} \qquad \text{and} \qquad \frac{\partial i_b}{\partial v_c} = g_m = \frac{\mu}{r_p} \tag{35}$$

For present purposes, assume that μ varies only slightly over the operating range of the tube and may thus be considered constant. Under these conditions, and since $\mu = \partial v_b / \partial v_c$, any partial derivative of the form $\partial^p i_b / \partial v_b^{p-q}\, \partial v_c{}^q$ takes the form

$$\frac{\partial^p i_b}{\partial v_b^{p-q}\, \partial v_c{}^q} = \left(\frac{\partial v_b}{\partial v_c} \right)^q \frac{\partial^p i_b}{\partial v_b{}^p} = \mu^q \frac{\partial^p i_b}{\partial v_b{}^p} \tag{36}$$

which will be used to evaluate some of the terms of (33).

Evaluating further,

$$\frac{\partial^2 i_b}{\partial v_b{}^2} = \frac{\partial}{\partial v_b} \left(\frac{1}{r_p} \right) = -\frac{1}{r_p{}^2} \frac{\partial r_p}{\partial v_b} \tag{37}$$

Equations (36) and (37) make

$$\frac{\partial^2 i_b}{\partial v_b\, \partial v_c} = \mu \frac{\partial^2 i_b}{\partial v_b{}^2} = -\frac{\mu}{r_p{}^2} \frac{\partial r_p}{\partial v_b}$$

and

$$\frac{\partial^2 i_b}{\partial v_c{}^2} = \mu^2 \frac{\partial^2 i_b}{\partial v_b{}^2} = -\frac{\mu^2}{r_p{}^2} \frac{\partial r_p}{\partial v_b} \tag{38}$$

By similar processes,

$$\begin{aligned} \frac{\partial^3 i_b}{\partial v_b{}^3} &= \frac{2}{r_p{}^3} \left(\frac{\partial r_p}{\partial v_b} \right)^2 - \frac{1}{r_p{}^2} \frac{\partial^2 r_p}{\partial v_b{}^2} \\ \frac{\partial^3 i_b}{\partial v_b\, \partial v_c{}^2} &= \mu^2 \left[\frac{2}{r_p{}^3} \left(\frac{\partial r_p}{\partial v_b} \right)^2 - \frac{1}{r_p{}^2} \frac{\partial^2 r_p}{\partial v_b{}^2} \right] \\ \frac{\partial^3 i_b}{\partial v_b{}^2\, \partial v_c} &= \mu \left[\frac{2}{r_p{}^3} \left(\frac{\partial r_p}{\partial v_b} \right)^2 - \frac{1}{r_p{}^2} \frac{\partial^2 r_p}{\partial v_b{}^2} \right] \\ \frac{\partial^3 i_b}{\partial v_c{}^3} &= \mu^3 \left[\frac{2}{r_p{}^3} \left(\frac{\partial r_p}{\partial v_b} \right)^2 - \frac{1}{r_p{}^2} \frac{\partial^2 r_p}{\partial v_b{}^2} \right] \end{aligned} \tag{39}$$

Substituting (34), (35), and (37) to (39) into (33) and collecting like terms, we have

$$i_b = I_{b0} + \frac{v_p + \mu v_g}{r_p} - \frac{(v_p + \mu v_g)^2}{2!r_p^2}\frac{\partial r_p}{\partial v_b} + \frac{(v_p + \mu v_g)^3}{3!r_p^3}\left[2\left(\frac{\partial r_p}{\partial v_b}\right)^2 - r_p\frac{\partial^2 r_p}{\partial v_b^2}\right] + \cdots \quad (40)$$

It is to be noted that the partial derivatives are to be evaluated where the plate voltage is V_{b0} and where the grid voltage is V_{c0}. While the above result is not widely applicable to numerical calculations, it is useful in the theoretical study of electron tubes.

PROBLEMS

1. Obtain a Taylor's series for ln $(2 + h)$ in powers of h.

2. Obtain a Taylor's series for sin $(\pi/6 + h)$ in powers of h.

3. Derive a Taylor's series for cosh $(a + h)$ in powers of h.

4. Express tan $(\pi/4 + h)$ as a Taylor's series in powers of h.

5. Two tubes (assumed to have identical characteristics) are operated in push-pull with their plates connected to a transformer primary, so that the magnetizing effect in the core is proportional to the *difference* between their plate currents. The current i_{b1} in tube 1 can be expressed as a Taylor's series in powers of v_g, as in (23). The constants A, B, etc., are determined by the tube characteristics and operating voltages. In tube 2, the plate current i_{b2} is represented by a similar series, except that the values are the negatives of those in (23). (*a*) Write the series for the current in tube 2. (*b*) Write the difference between these series, showing that in the output circuit the even-order harmonics are balanced out (see last sentence of Example 2, Sec. 18-5).

6. Express sin (xy) as a Taylor's series of the form (29), for the case where $x = 1$ and $y = \pi/2$. Include terms as far as those in the second pair of brackets.

7. Express cosh $(x + y)$ as a Taylor's series of the form (29). Include terms in the third pair of brackets.

8. The impedance of a series RL circuit is $Z = (X^2 + R^2)^{1/2}$. Express as a Taylor's series of form (29), including terms in the third pair of brackets.

REFERENCES

1. F. L. Griffin: "Mathematical Analysis: Higher Course," pp. 307–315, Houghton Mifflin Company, Boston, 1927.
2. E. Goursat: "A Course in Mathematical Analysis," trans. E. R. Hedrick, 2d ed., vol. I, pp. 107–108, Ginn & Company, Boston, 1904.
3. C. R. Wylie: "Calculus," p. 451, McGraw-Hill Book Company, Inc., New York, 1953.

19

Fourier Series

In the preceding two chapters we found that many useful functions can be expanded in power series. Here we take up a different kind of series, a *trigonometric* series.

19-1 Fourier series. It is commonly known that a complex or distorted wave, like many of the electric waves we encounter in practice, can be considered as the sum of

1. An *average* (or dc) value.
2. A *fundamental* sinusoidal wave of the same frequency as the given wave.
3. And a series of sinusoidal *harmonics*, whose frequencies are whole-number multiples of the frequency of the given wave.

Our problem is to find the relative amounts of dc component, fundamental, and various harmonics that may be present in a given waveform. This is called *analyzing* the waveform.

Let $f(t)$ indicate the distorted wave we want to analyze. As a first step, it will later appear convenient to let $a_0/2$ represent the average, or dc, value of the wave.

Before continuing with the analysis, we have to decide which instant we shall consider to be the beginning of a cycle of the waveform and

designate this instant as $t = 0$. This matter is somewhat arbitrary. There are some advantages in selecting $t = 0$ in certain ways, as described later in this chapter. For the moment, let us select $t = 0$ as an instant when the given waveform goes through a zero value in the positive direction.

Let the given waveform have a frequency f cycles per second. The fundamental component will be a sine wave whose frequency is also f, so that its angular frequency is $\omega = 2\pi f$. Let the amplitude of the fundamental component be c_1. Then if the fundamental goes through its zero value in a positive direction at the instant selected as $t = 0$, the fundamental component can be expressed as

$$c_1 \sin \omega t$$

On the other hand, the fundamental might lead or lag this position by some angle θ_1 (if θ_1 is *positive,* this indicates a *leading* fundamental component). Then the fundamental would be given by

$$c_1 \sin (\omega t + \theta_1)$$

Such a leading or lagging fundamental can be resolved into two components:

1. A *quadrature component* that goes through its zero value in a positive direction at an instant 90° removed from the instant $t = 0$. Let the peak value of this quadrature component be a_1. Then this component is expressed by

$$a_1 \cos \omega t$$

2. An *in-phase component* that goes through its zero value in a positive direction when $t = 0$. Let the peak value of the in-phase component be b_1. Then this component is expressed as

$$b_1 \sin \omega t$$

Thus a leading or lagging fundamental is expressed as the sum of the quadrature and the in-phase components:

$$c_1 \sin (\omega t + \theta_1) = a_1 \cos \omega t + b_1 \sin \omega t$$

Since the quadrature and the in-phase components are separated by 90°,

$$c_1^2 = a_1^2 + b_1^2$$

and

$$\tan \theta_1 = \frac{b_1}{a_1}$$

By giving suitable values to a_1 and b_1, we can represent a fundamental wave whose phase angle and amplitude correspond with those of the actual fundamental present in the wave being considered.

A *second harmonic* wave, if one is present, would have an angular frequency 2ω. If it goes through its zero value in a positive direction when $t = 0$, it is expressed as

$$c_2 \sin 2\omega t$$

where c_2 is its peak value. But if it leads or lags this position by some angle θ_2 (measured at the second-harmonic frequency), then the second-harmonic component is expressed as

$$c_2 \sin (2\omega t + \theta_2)$$

and this leading or lagging second-harmonic component can be resolved into quadrature and in-phase components:

$$c_2 \sin (2\omega t + \theta_2) = a_2 \cos 2\omega t + b_2 \sin 2\omega t$$

in a manner similar to that for the fundamental. Here

$$c_2{}^2 = a_2{}^2 + b_2{}^2$$

and

$$\tan \theta_2 = \frac{b_2}{a_2}$$

Similarly for other harmonics if present.

Thus our distorted wave can be represented by

$$f(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + \cdots$$
$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + \cdots \quad \textbf{(1)}$$

This expression is a *Fourier series** representing the given wave $f(t)$. If, for a given wave, we are able to evaluate the constant coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, we can state the strengths of the various dc, fundamental, and harmonic components of the wave. We may be given the original wave in any of three ways: as an equation, as a table of the values taken by the wave at various time intervals through its cycle, or as a graph such as an oscillogram. We shall first treat the case in which the given wave is expressed as an equation.

19-2 Formulas for the Fourier coefficients. The analysis of a complex wave consists of finding the numerical values of the a's and b's of the series (1) corresponding to the given wave. This information

* Named for a pioneer worker with this series, J. B. J. Fourier of France. The principal publication of his results appeared in 1812. Another expression of the Fourier series is

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\omega t + b_m \sin m\omega t)$$

tells us what relative amounts of dc, fundamental, and the various harmonics go to make up the wave.

We now obtain some formulas by which we can determine these coefficients. The *use* of these formulas will not, in general, prove difficult.

a. Finding a_0. Let us integrate the expression (1), term by term, with respect to ωt. Carrying out this integration over the range of 1 cycle, that is, between the limits $\omega t = 0$ and $\omega t = 2\pi$, we get

$$\int_0^{2\pi} f(t)\, d(\omega t) = \int_0^{2\pi} \frac{a_0}{2}\, d(\omega t) + \text{remainder} \tag{2}$$

Here the remainder consists of the sum of the integrals of all the cosine and sine terms of (1). If you evaluate any one of these terms, such as

$$\int_0^{2\pi} a_1 \cos \omega t\, d(\omega t)$$

you will have little trouble convincing yourself that each of these integrals in the remainder equals zero. In fact, the effect is that of integrating sinusoidal waves over a whole number of cycles, which we have already seen gives the result zero (Sec. 11-14, Example 1).

We therefore put the remainder in (2) equal to zero. Carrying out the integration of the right member of (2) and solving the result for a_0, we get

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t)\, d(\omega t) \tag{3}$$

b. Finding a_m. Now we derive a formula for any of the a's other than a_0. That is, we are after the a coefficient for some particular harmonic, say the mth.

We multiply the terms of (1) by $\cos m\omega t\, d(\omega t)$ and integrate each between the limits $\omega t = 0$ and $\omega t = 2\pi$:

$$\int_0^{2\pi} f(t) \cos m\omega t\, d(\omega t) = \int_0^{2\pi} \frac{a_0}{2} \cos m\omega t\, d(\omega t) + \text{remainder} \tag{4}$$

The first integral in the right member is equal to zero. The remainder terms are all of either of the forms

$$\int_0^{2\pi} a_n \cos m\omega t \cos n\omega t\, d(\omega t) \tag{5}$$

or

$$\int_0^{2\pi} b_n \cos m\omega t \sin n\omega t\, d(\omega t) \tag{6}$$

where n represents some subscript (1, 2, etc.) of a or b.

The form (5) may be integrated by getting the indefinite integral from tables and inserting the limits, but this is hardly necessary. A trigonometric identity shows that

$$\cos m\omega t \cos n\omega t = \tfrac{1}{2}[\cos (m + n)\omega t + \cos (m - n)\omega t] \tag{7}$$

This reduces (5) to the sum of the integrals of two separate cosine terms. Since m and n are the *orders of harmonics,* they are whole numbers; and the integration between the limits $\omega t = 0$ and $\omega t = 2\pi$ is equivalent to integrating cosine waves over a whole number of cycles. Thus, the value of each term in the remainder having the form (5) is zero—*with one exception.*

There will be one coefficient a_n having a subscript n equal to the order m of the harmonic with which we are working. For this term $m = n$. But when $m = n$, the second term in the right member of (7) becomes $\tfrac{1}{2} \cos 0 = \tfrac{1}{2}$, a constant. Integrated according to (5), it gives πa_m; this is the only result (other than zero) for terms of the form (5).

Turning to (6), we replace $\cos m\omega t \sin n\omega t$ with its identity $\tfrac{1}{2}[\sin (m + n)\omega t + \sin (n - m)\omega t]$. This makes all integrals of the form (6) equal to zero—even when $m = n$, for then the last term becomes $\tfrac{1}{2} \sin 0$. This makes (4)

$$\int_0^{2\pi} f(t) \cos m\omega t \, d(\omega t) = a_m\pi$$

or

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos m\omega t \, d(\omega t) \tag{8}$$

c. Finding b_m. To get a formula for b_m the series (1) is multiplied by $\sin m\omega t \, d(\omega t)$, then integrated between $\omega t = 0$ and $\omega t = 2\pi$. Methods similar to those used in finding a_m then yield

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin m\omega t \, d(\omega t) \tag{9}$$

Formulas (3), (8), and (9) are called the *Euler formulas.*

19-3 An example of Fourier analysis. We now analyze, as an example, the sawtooth wave

$$i = \omega t \tag{10}$$

pictured in Fig. 19-1. First, we find the average, or dc, value of the wave. From (3),

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \omega t \, d(\omega t) = 2\pi \qquad \text{or} \qquad \frac{a_0}{2} = \pi \tag{11}$$

(In this case our procedure can be confirmed by finding the area of one triangle of the wave and dividing it by the length of the base.)

The coefficient of the cosine term for any harmonic, say the mth, is by (8)

$$a_m = \frac{1}{\pi} \int_0^{2\pi} \omega t \cos m\omega t \, d(\omega t)$$

You can carry out this integration or, better, obtain it from tables:

$$\begin{aligned} a_m &= \frac{1}{\pi} \left(\frac{1}{m^2} \cos m\omega t + \frac{1}{m} \omega t \sin m\omega t \right) \Bigg]_{\omega t=0}^{2\pi} \\ &= \frac{1}{\pi} \left[\frac{1}{m^2} \cos 2m\pi + \frac{1}{m} (2\pi) \sin 2m\pi - \frac{1}{m^2} \cos 0 - \frac{1}{m} (0) \sin 0 \right] \end{aligned}$$

Since m is a whole number, the above expression takes the value zero. Thus, this particular wave has no cosine terms.

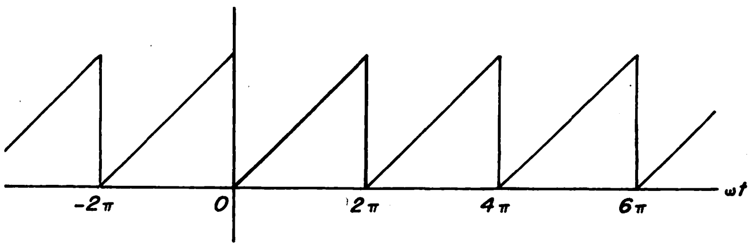


Fig. 19-1

The coefficient of the sine term of any harmonic is by (9)

$$\begin{aligned} b_m &= \frac{1}{\pi} \int_0^{2\pi} \omega t \sin m\omega t \, d(\omega t) = \frac{1}{\pi} \left(\frac{1}{m^2} \sin m\omega t - \frac{1}{m} \omega t \cos m\omega t \right) \Bigg]_{\omega t=0}^{2\pi} \\ &= \frac{1}{\pi} \left[\frac{1}{m^2} \sin 2m\pi - \frac{1}{m} (2\pi) \cos 2m\pi - \frac{1}{m^2} \sin 0 + \frac{1}{m} (0) \cos 0 \right] \end{aligned}$$

Since m is a whole number, this becomes

$$b_m = -\frac{2}{m} \tag{12}$$

Inserting the values found above in (11) and (12) for a_0 and for b_m into (1), we note that $m = 1$ for the first b term, $m = 2$ for the second b term, etc.; that is, m is the order of the harmonic under consideration in any one term:

$$i = \pi - 2(\sin \omega t + \tfrac{1}{2} \sin 2\omega t + \tfrac{1}{3} \sin 3\omega t + \tfrac{1}{4} \sin 4\omega t + \cdots) \tag{13}$$

The minus sign indicates a negative polarity for the various sine waves; that is, these waves go through their zero values in a negative

direction at the point where $\omega t = 0$. The first few terms of this series are plotted in Fig. 19-2, showing how they add to give approximately the sawtooth wave. The accuracy of the series in giving the original function increases as more and more terms are used.

➢ It is important in deriving a series in the above manner to note whether a_m or b_m changes form according to whether m is an even or an odd number.

Sometimes the cosine or the sine terms are present for *only* the even (or *only* the odd) harmonics.

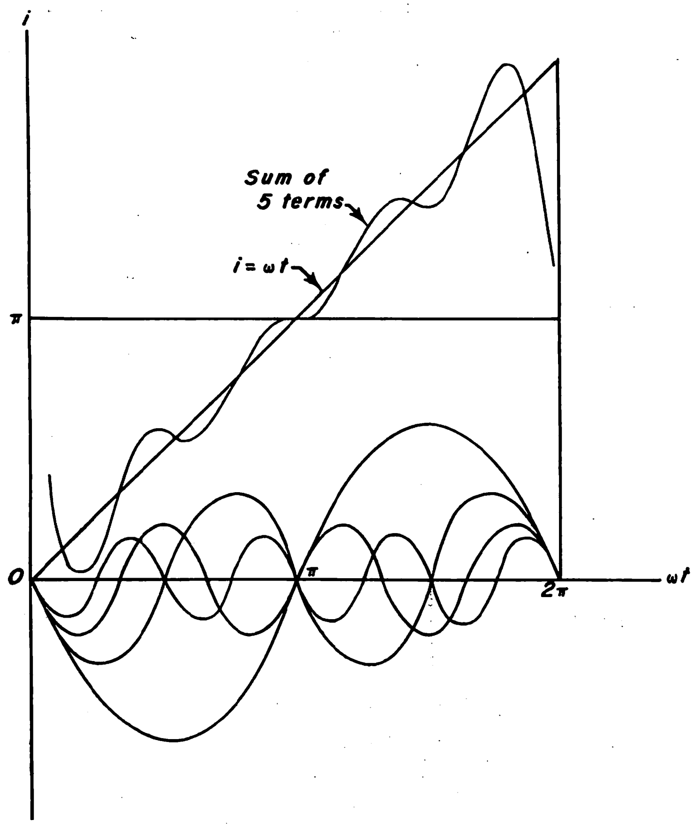


Fig. 19-2

If the peak value of the wave is made other than 2π units, the series (13) can be adapted to the new wave simply by a change in scale. If, for instance, the peak value is 1 unit, then each term of (13) should be divided by 2π.

We can find from the above series the percentage of any harmonic

relative to the fundamental. For instance, the fourth harmonic has an amplitude of $\frac{1}{2}$, while the fundamental has an amplitude of 2 (in current units, perhaps amperes). Therefore

$$\text{Per cent fourth harmonic} = \frac{\frac{1}{2}}{2} \times 100 \text{ per cent} = 25 \text{ per cent}$$

QUESTIONS

1. Give a general form for a Fourier series for $f(t)$ valid from $\omega t = 0$ to $\omega t = 2\pi$.
2. State the information contained in the term $a_0/2$ in the Fourier series.
3. State the formulas for the Fourier coefficients a_0, a_m, and b_m.
4. Having obtained the numerical values of a_3 and b_3, how would we get the amplitude c_3 of the third-harmonic component of a wave?

PROBLEMS

1. Find from (13) the percentage of second-harmonic component in the given sawtooth wave, compared with the fundamental component.

2. Write three more terms for (13) further describing the harmonic structure of the sawtooth wave $i = \omega t$.

3. Derive a Fourier series valid in the interval from $\omega t = 0$ to $\omega t = 2\pi$ for an exponential voltage wave $v = e^{\omega t}$. Get four cosine terms and four sine terms, as well as the dc value.

4. In some automatic-frequency-control circuits for the horizontal-deflection systems of television receivers, use is made of a parabolic wave, $v = (\omega t)^2$. (*a*) Derive a Fourier series representing this wave in the interval from $\omega t = 0$ to $\omega t = 2\pi$. (*b*) Find the percentage of second harmonic, referred to the fundamental. (*c*) Find the percentage of third harmonic, referred to the fundamental.

5. The output waveform of a full-wave rectifier without filter consists of a succession of half cycles of sinusoidal form, all of the same polarity. This may be described by the equation $v = V_{\max} \sin (\omega t/2)$, where $\omega/2$ is the angular frequency of the original ac wave. Describe this wave by means of a Fourier series which is valid in the interval from $\omega t = 0$ to $\omega t = 2\pi$.

19-4 Existence of Fourier series. In beginning studies of electricity we learn without proof that any periodic distorted or nonsinusoidal electric wave can be considered as the sum of a dc component, a sinusoidal fundamental wave of the frequency of the given wave, and a series of sinusoidal harmonics whose frequencies are integral multiples of the frequency of the given wave. We may even become so accustomed to this assumption that we sometimes fail to wonder whether or not it can be proved.

To prove that any periodic wave which we are likely to encounter in practice can be thus represented is beyond the scope of this book. We shall content ourselves with the statement that

A periodic function of t may be represented in the interval from $\omega t = 0$ to $\omega t = 2\pi$ by a series of the form (1) if the function (*a*) is single-valued, (*b*) has only a finite number of maxima, and (*c*) is continuous, or has at most a finite number of finite discontinuities.[1]

A formal treatment of Fourier series would *prove* that waves represented by these functions can be represented by the series. It would also attempt to show what *other* forms of waves might have Fourier series; this problem has not been completely solved, even though some of the greatest mathematicians have worked upon it. Also, we have not proved here that a correct result is obtained from the integration, term

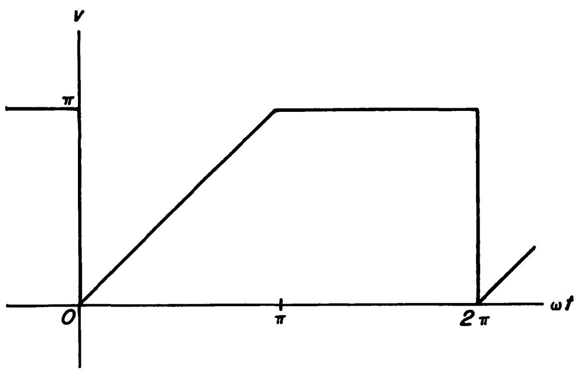


Fig. 19-3

by term, of the *infinite* trigonometric series (1) in deriving the Euler formulas (3), (8), and (9).[1,2]

The Fourier series is an outstanding result of mathematical analysis. It has opened the way for a tremendous amount of practical and theoretical work which cannot be mentioned here. Some questions have yet to be answered by mathematicians before the full possibilities of this vast subject are realized.

19-5 Intervals other than $(0,2\pi)$. We have assumed thus far that a function being analyzed conforms to its given equation over a range of values from $\omega t = 0$ to $\omega t = 2\pi$. But it is often desirable to use other intervals. For example, the interval from $\omega t = -\pi$ to $\omega t = \pi$ is often used. The theorem of the previous section may be restated, using these limits, and Formulas (3), (8), and (9) may be used as they are except for the insertion of the new limits $-\pi$ and π in place of 0 and 2π. The resulting series for a given wave is changed in appearance, but it contains the same information except for the assumed starting point of the cycle.

19-6 Broken intervals. Sometimes a wave does not follow a single equation throughout its entire cycle. For instance, the clipped saw-

tooth wave of Fig. 19-3 has the equation

$$v = \omega t \tag{14a}$$

from $\omega t = 0$ to $\omega t = \pi$, but takes the form

$$v = \pi \tag{14b}$$

when ωt lies between π and 2π. To analyze such a wave the procedure is to split the integrals (3), (8), and (9) into separate integrals covering the individual parts of the cycle, through a proper choice of limits:

$$a_0 = \frac{1}{\pi}\int_0^\pi \omega t\, d(\omega t) + \frac{1}{\pi}\int_\pi^{2\pi} \pi\, d(\omega t) \tag{15}$$

$$a_m = \frac{1}{\pi}\int_0^\pi \omega t \cos m\omega t\, d(\omega t) + \frac{1}{\pi}\int_\pi^{2\pi} \pi \cos m\omega t\, d(\omega t) \tag{16}$$

$$b_m = \frac{1}{\pi}\int_0^\pi \omega t \sin m\omega t\, d(\omega t) + \frac{1}{\pi}\int_\pi^{2\pi} \pi \sin m\omega t\, d(\omega t) \tag{17}$$

PROBLEMS

1. Evaluate (15) to (17) and write a Fourier series for the wave of Fig. 19-3 valid from $\omega t = 0$ to $\omega t = 2\pi$.

2. A square-wave generator provided an output of π volts from $\omega t = 0$ to $\omega t = \pi$ and an output of 0 volts from $\omega t = \pi$ to $\omega t = 2\pi$. Write a Fourier series valid from $\omega t = 0$ to $\omega t = 2\pi$ for the output of this generator.

3. A circuit generated a periodic wave which followed the formula $v = (\omega t)^2$ throughout its cycle. This wave was then sent through a clipper which clipped the wave at a level of π^2 volts. Write a Fourier series valid from $\omega t = 0$ to $\omega t = 2\pi$ representing this waveform.

4. A filterless half-wave rectifier provides an output $v = \sin \omega t$ from $\omega t = 0$ to $\omega t = \pi$. The output is zero from $\omega t = \pi$ to $\omega t = 2\pi$. Analyze the output waveform to obtain a Fourier series applicable from $\omega t = 0$ to $\omega t = 2\pi$.

5. A current wave rises according to $i = \omega t$, during the interval from $\omega t = 0$ to $\omega t = \pi$. From $\omega t = \pi$ to $\omega t = 2\pi$, the wave decreases according to $i = 2\pi - \omega t$. Derive a Fourier series describing the wave from $\omega t = 0$ to $\omega t = 2\pi$.

19-7 Analyzing tables and graphs. Instead of having a *formula* for the function to be analyzed, we may have it given as a table of measured values or as a graph, an oscillogram, for instance. The principles which have already been laid out enable us to analyze the function, at least approximately, in these cases.

Let us take as an example the function of Fig. 19-4. The first step is to divide the horizontal axis for the interval of one cycle into a number of segments. For accurate results the axis should be cut into no fewer than about 12 segments. Also, the number of segments must be greater than twice the order of the highest harmonic which it is desired to evaluate. In our example, we shall use 20 equal segments. Next, perpen-

diculars are erected at the ends of the segments. The result is to make 20 rectangles.

Now we can get the coefficients of the various cosine and sine terms.

➢ We can evaluate any desired term of the series without having to find the previous terms first.

The procedure includes making a schedule, or table, for each term desired.

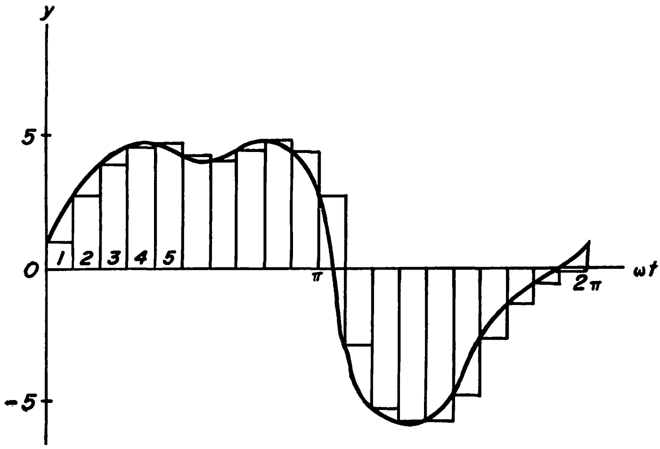


Fig. 19-4

Each of these tables takes the place of one integration according to Formula (8) or (9). As an example, let us find b_3, the coefficient of the sine term for the third-harmonic content of the wave of Fig. 19-4. The calculations are made according to Table 19-1.*

* We are trying to get the value of b_3 corresponding to Formula (9):

$$b_3 = (1/\pi) \int_0^{2\pi} f(t) \sin 3\omega t \, d(\omega t)$$

It will be remembered that a definite integral of a function can be had by getting the area under its graph. This area is approximately that of the sum of the individual small rectangles, like those constructed in Fig. 19-4. Each rectangle has an area equal to its height multiplied by its base length, the base here being $\pi/10$. Our problem, however, is not to get the integral of $f(t)$ itself, but rather the integral of the product $f(t) \sin 3\omega t$. To accomplish this it is necessary before summing the areas of the rectangles to multiply the area of each by the sine of three times its angular position. (Here, we use 18° steps, each step being the equivalent of $\pi/10$ radian.)

If we were to carry out the work in minute detail, we should enter in column 5 a quantity different from that actually used. This quantity would be the actual *area* of each rectangle—its height times the base $\pi/10$—multiplied by $\sin 3\omega t$. Then the sum of the entries in such a column would give approximately the integral for b_3. Multiplying by $1/\pi$, we would get the value of b_3. But the effects of multiplying by π and of dividing by π cancel each other, so that we use the shorter process given in the text.

The entries in column 1 simply identify the rectangles by number. Each entry in column 2 establishes the position of a rectangle (ωt degrees from the origin). In column 3 we record the measured height of each rectangle. In column 4 are entered the values of $\sin m\omega t$ or $\cos m\omega t$ corresponding to the coefficient we are working for. Since we are now solving for b_3, the entries are the values of $\sin 3\omega t$.

Now, for each rectangle, we find the product of the height and the corresponding quantity $\sin 3\omega t$ and enter the product in column 5. Next,

Table 19-1 Evaluating the Coefficient b_3 of the Wave of Fig. 19-4

(1) Number	(2) ωt, degrees	(3) $f(t)$	(4) $\sin 3\omega t$	(5) (3) × (4)
1	0	1.1	0.000	0.00
2	18	3.0	0.809	2.43
3	36	4.0	0.951	3.80
4	54	4.6	0.309	1.42
5	72	4.7	−0.588	−2.76
6	90	4.3	−1.000	−4.30
7	108	4.0	−0.588	−2.35
8	126	4.4	0.309	1.36
9	144	4.7	0.951	4.47
10	162	4.3	0.809	3.48
11	180	3.0	0.000	0.00
12	198	−3.0	−0.809	2.43
13	216	−5.2	−0.951	4.95
14	234	−5.9	−0.309	1.82
15	252	−5.9	0.588	−3.47
16	270	−5.0	1.000	−5.00
17	288	−3.0	0.588	−1.76
18	306	−1.7	−0.309	0.53
19	324	−0.7	−0.951	0.67
20	342	−0.2	−0.809	0.16
S				7.88

we get the sum S of the quantities of column 5. This sum is 7.88. Finally, the coefficient b_3 is obtained by dividing S by one-half the number of rectangles used. Here we divide S by 10, getting $b_3 = 0.788$.

To get the constant $a_0/2$, the average value of the function, we obtain approximately the average height of the curve. Here this is one-twentieth of the sum of the perpendiculars. Adding the entries of column 3 and dividing the sum by 20, we have $a_0/2 = 0.575$.

Further coefficients are obtained by making additional tables. To get a_3, for instance, we would enter in column 4 the values of $\cos 3\omega t$. After one table has been made, it is not essential to repeat the second and third columns in the additional tables.

If we are given a table of values, rather than a graph, we need only write the given values into column 3 of our table, without having to make measurements on a graph to get them.

We have noted (Sec. 19-4) that a Fourier series can be used *to display a wave having an abrupt jump or a sharp corner*. Such a wave cannot be represented by a Maclaurin's series or by a Taylor's series. It may be shown that, at a point where a function takes an abrupt jump, a Fourier series yields a value for the function equal to the average of the two

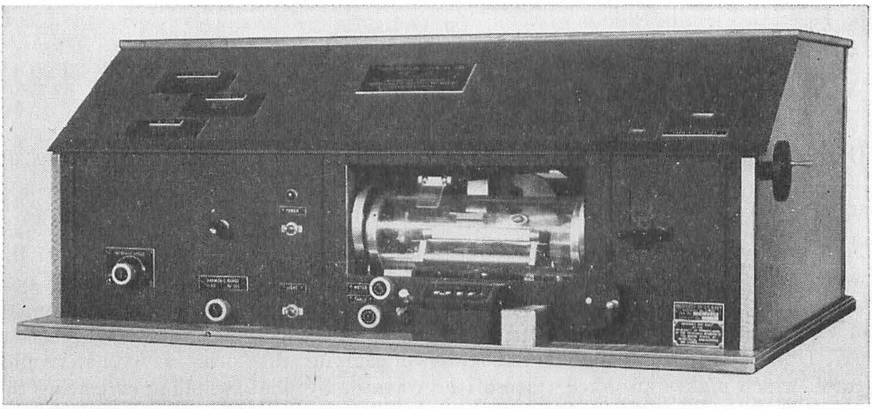


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Fig. 19-5

extremes of the jump. If a function makes a jump at the junction of two cycles, the first entry in column 3 of the schedule should be the *average* of the starting value and the ending value.

Machines have been constructed to carry out the integrations needed to evaluate the Fourier coefficients. Figure 19-5* shows a machine which mechanically performs the required integrations and gives directly the Fourier coefficients for the first 100 harmonics of any wave we might expect to encounter in practice.

PROBLEMS

1. (*a*) Find the coefficient a_3 of the cosine term of the third-harmonic component of the wave of Fig. 19-4 as determined from data in Table 19-1. (*b*) Find the total third-harmonic content of the wave.

2. Determine the approximate amount of second-harmonic component of the wave of Fig. 19-4 expressed as a percentage of the fundamental component.

3. Find the total second-harmonic component of the class C amplifier plate-current wave whose oscillogram is shown in Fig. 19-6.

* This figure illustrates work performed on a U.S. Navy project in the Moore School of the University of Pennsylvania.

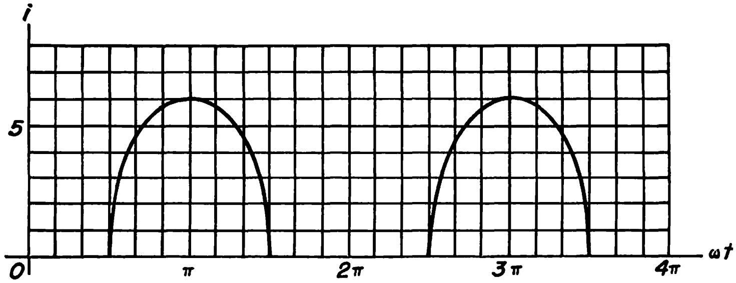


Fig. 19-6

4. Find the amount of third-harmonic component in the wave of Fig. 19-6 expressed as a percentage of the fundamental component.

5. A periodic voltage wave has instantaneous values as follows:

Degrees............	0	30	60	90	120	150	180	210	240	270	300	330
Volts...............	6.1	7.4	10.2	3.9	−6.2	−10.8	−14.0	−11.0	−5.8	4.0	9.5	7.5

Find the coefficient a_2 of the cosine term of the second-harmonic component of the wave.

6. Find the approximate percentage of the third-harmonic component of the wave of Prob. 5 expressed as a percentage of the fundamental.

7. Obtain one or more actual oscillograms of complex waveforms, as with an oscillograph which makes an inked trace of the wave or by photographing or tracing the waveform from the screen of a cathode-ray oscilloscope. Determine the amounts of dc, fundamental, second-harmonic, and third-harmonic components in these waves.

19-8 Symmetry in a graph. We can tell much about the harmonic composition of a wave by its characteristics of *symmetry*. As we shall

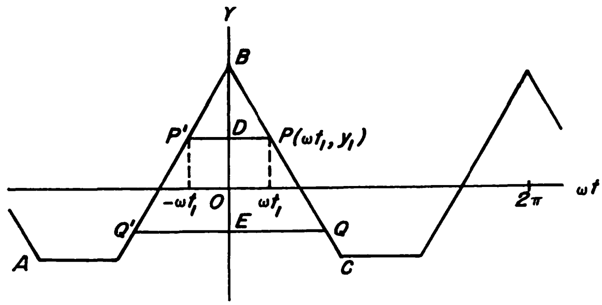


Fig. 19-7

use it here, symmetry in a graph is an indication of the *correspondence* of the various points along sections of the graph. For instance, a plane-mirror image is said to be symmetrical with the object which produces it. Similarly, the two gloves of a pair are symmetrical.

a. Symmetry about the vertical axis. The wave ABC of Fig. 19-7 is said to be symmetrical with respect to the vertical axis OY. Let us see what is meant by this statement.

If we draw a straight line, like $P'P$ or $Q'Q$, between any two corresponding points on the two sections of ABC, then, for the wave shown, OY will cut this straight line into equal parts. Thus $P'D = DP$, and $Q'E = EQ$.

➽ When a line (in this case OY) divides into equal parts the straight lines connecting corresponding points on two sections of a curve, the curve is said to be symmetrical with respect to the line.

Let us see what this symmetry tells us. Consider the point $P(\omega t_1, y_1)$. For a symmetry of the kind shown in Fig. 19-7 to exist, it is necessary that the wave take this same value y_1 when ωt is made equal to $-\omega t_1$. But in order that this condition may apply, the Fourier series for the

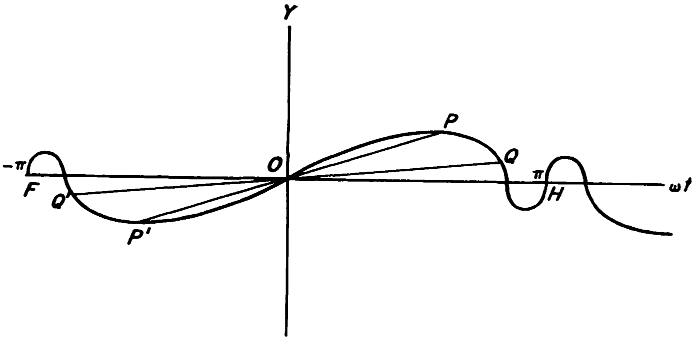


Fig. 19-8

wave cannot contain any sine terms. For if any terms like sin ωt or sin $2\omega t$ are present, they will have algebraic signs which are opposite when ωt is negative to the signs obtaining when ωt is positive, and they will therefore make the wave take values, for negative values of ωt, which are different from those obtained for positive values of ωt. Cosine terms, on the other hand, have the same algebraic sign for negative ωt as they do for similar positive values of ωt, so that they do not destroy this type of symmetry (compare Probs. 3 and 4, Sec. 19-7).

If a function $f(x)$ has the property that $f(-x) = f(x)$, we call this an *even* function. Figure 19-7 is, therefore, the graph of an even function; and we say that a wave represented by an even function can have no sine terms in its Fourier expansion.

b. Symmetry about the origin. In the wave of Fig. 19-8, the origin O divides into equal parts any straight lines like $P'P$ or $Q'Q$ which connect corresponding points on the two sections of the wave FOH. In this case, the wave is not symmetrical about a line, but rather about a *point*, the origin.

A graph like Fig. 19-8 contains sine terms only, for its two sections FO and OH are the negatives of each other. Any cosine terms would intro-

duce values destroying this symmetry. We note, then, that if the graph of a wave is symmetrical about the origin, there can be no cosine terms in its Fourier series. The cosine terms correspondingly will be absent if the *formula* for the wave is changed in sign, but unchanged otherwise, when ωt is changed to $-\omega t$.

A function $f(x)$ having the property that $f(-x) = -f(x)$ is called an *odd* function. The graph of Fig. 19-8 is thus a graph of an odd function;

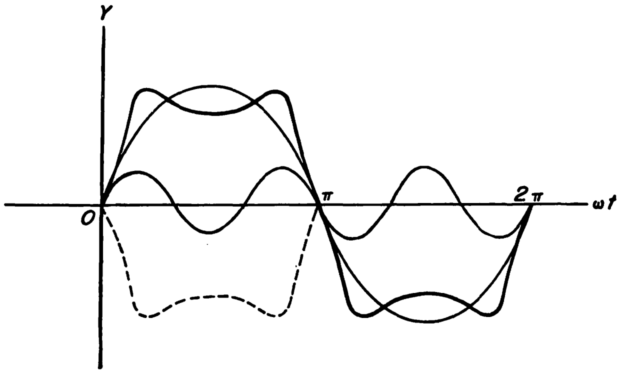


Fig. 19-9

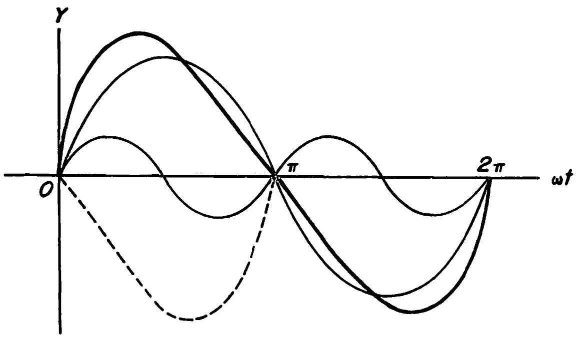


Fig. 19-10

and we say that a wave represented by an odd function can have no cosine terms in its Fourier expansion.

c. Symmetry about the horizontal axis. If a periodic wave contains no even harmonics, it is symmetrical about the horizontal axis (or some horizontal line) in the sense that if we displace the lower half of the wave by $\frac{1}{2}$ cycle, it is then the mirror image of the upper portion of the wave. A wave of this kind is shown in Fig. 19-9. In this case the wave consists of a fundamental and a third harmonic.

If there are any even harmonics in the wave, it cannot have this kind of symmetry. This fact is presented in Fig. 19-10, for the case of a fundamental and a second harmonic.

QUESTIONS

1. What information becomes known about the Fourier series for a function if its graph is symmetrical about the vertical axis?

2. A given wave is symmetrical about the origin. What information does this fact supply about the Fourier series representing the wave?

3. If a wave is symmetrical about the horizontal axis, in the sense that the lower half of the wave is the mirror image of the upper half, but displaced by ½ cycle, what kind of harmonics must the wave contain?

PROBLEMS

1. By inspection of Fig. 19-11 find with respect to what lines or points the various waves are symmetrical.

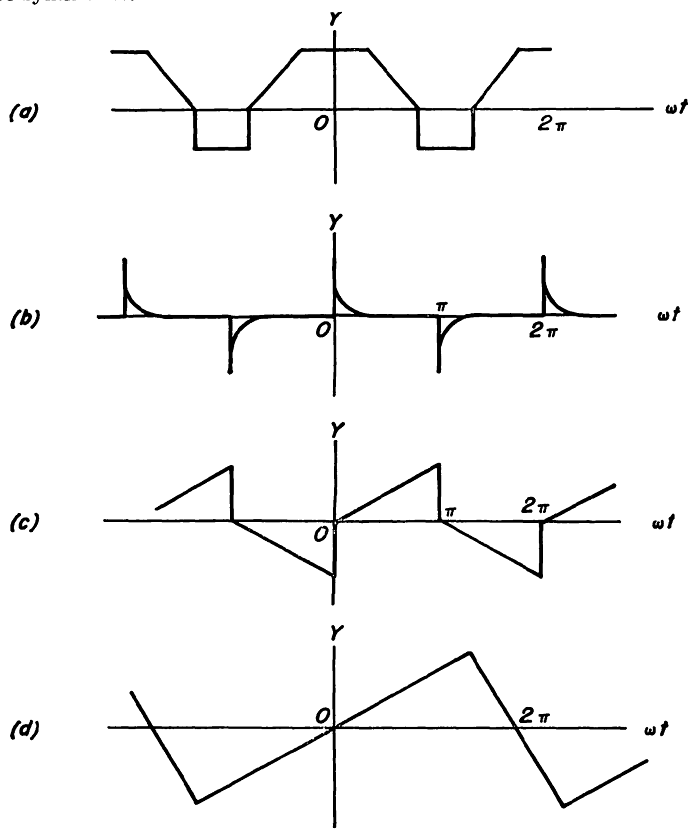


Fig. 19-11

2. Referring to Prob. 1, state the information about the Fourier series for the various given waves which can be gained from their properties of symmetry.

3. In Fig. 19-12 what change should be made in the location of the axes of each of the waves to simplify the Fourier series for the wave?

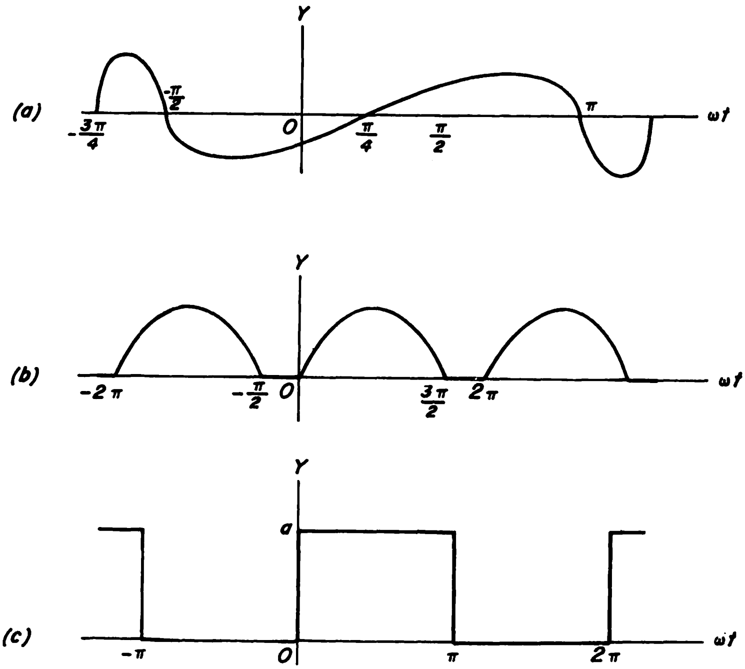


Fig. 19-12

4. Determine which of the following functions are even and which are odd. Which contain no sine terms in their Fourier series? No cosine terms?

(a) $i = \omega t - 10$ (c) $q = (\omega t)^3$ (e) $i_b = \omega t - (\omega t)^2$
(b) $i = (\omega t)^2 - 10$ (d) $v = \sin^3 \omega t$ (f) $v = 1 - \sin \omega t$

19-9 Cosine and sine series. It is often possible to take a function whose Fourier series (in the interval, say, from $\omega t = 0$ to $\omega t = 2\pi$) contains both sine and cosine terms and revise it so that the resulting series is simpler—containing *only* sine terms or *only* cosine terms. The resulting series are valid, in general, *only* over the *half-cycle* interval from $\omega t = 0$ to $\omega t = \pi$, but they are useful notwithstanding this restriction.

Since the series now being taken up are especially useful in representing nonrecurring functions, we express them in terms of a general variable x, rather than ωt. The variable x may have as its physical significance such quantities as time t, voltage v, etc.

If a function $f(x)$ follows a given graph from $x = 0$ to $x = \pi$, we can make the graph symmetrical to the vertical axis if we *assign* suitable values to the function in the interval from $x = -\pi$ to $x = 0$. To meet this requirement we *assume* that from $-\pi$ to 0 the function $f(-x)$ is equal to $f(x)$, while from 0 to π the function takes its actual values. It may

be shown that, based upon these assumptions, most of the functions we commonly need can be represented by the cosine series

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \tag{18}$$

where the coefficients are calculated by

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx \tag{19}$$

and

$$a_m = \frac{2}{\pi} \int_0^{\pi} f(x) \cos mx\, dx \tag{20}$$

Observe that the external multiplying constants for the integrals in (19) and (20) are taken twice as large as those of our formulas applying to the entire cycle and that the integrations of (19) and (20) are correspondingly carried out over only ½ cycle.

A sine series can be derived in a similar way. Here we make the function appear symmetrical about the origin. We do this by assuming that in the range from $-\pi$ to 0 the function $f(-x)$ is equal to $-f(x)$. It can be shown that this results in

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \tag{21}$$

where

$$b_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin mx\, dx \tag{22}$$

Sometimes the sine series (21) is simpler than the cosine series (18) for a given function. Sometimes the reverse is true.

Example. A current i has a constant value $i = 1$ from time $t = 0$ to $t = \pi$. Express as a Fourier sine series in t.

Here $i = 1$ in the interval $t = 0$ to $t = \pi$. To obtain the sine series we assume that in the interval from $t = -\pi$ to $t = 0$ the current is $i = -1$. The Fourier coefficients become, by (22),

$$b_m = \frac{2}{\pi} \int_0^{\pi} \sin mt\, dt = -\frac{2}{m\pi} \cos mt \Big]_{t=0}^{\pi}$$

When m is an odd number, this gives $b_m = 4/m\pi$. When m is even, we get $b_m = 0$. The desired series is, by (21),

$$i = \frac{4}{\pi}\left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \cdots\right)$$

19-10 Series valid in the interval $(0,T)$. If a function $f(x)$ is of a kind expressible in a Fourier series in the interval from $x = 0$ to $x = T$, we may express the function as a Fourier cosine series. Briefly, the series can be shown to be

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{T} + a_2 \cos\frac{2\pi x}{T} + a_3 \cos\frac{3\pi x}{T} + \cdots \tag{23}$$

where
$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx \tag{24}$$

and
$$a_m = \frac{2}{T}\int_0^T f(x)\cos\frac{m\pi x}{T}\,dx \tag{25}$$

Or we may express the function as a Fourier sine series:

$$f(x) = b_1 \sin\frac{\pi x}{T} + b_2 \sin\frac{2\pi x}{T} + b_3 \sin\frac{3\pi x}{T} + \cdots \tag{26}$$

where
$$b_m = \frac{2}{T}\int_0^T f(x)\sin\frac{m\pi x}{T}\,dx \tag{27}$$

Example. A dc voltage is suddenly applied to an inductor of small resistance, so that approximately $i = (V/L)t$ (see Example 2 of Sec. 17-6). Write the current function as a Fourier cosine series valid in the interval from $t = 0$ to $t = T$.

Here the upper limit is the time T. To obtain a_0 we write, by (24),

$$a_0 = \frac{V}{L}\frac{2}{T}\int_0^T t\,dt = \frac{VT}{L}$$

To get a_m, we use (25):

$$a_m = \frac{V}{L}\frac{2}{T}\int_0^T t\cos\frac{m\pi t}{T}\,dt = \frac{2V}{LT}\left[\frac{T}{m\pi}t\sin\frac{m\pi}{T}t + \left(\frac{T}{m\pi}\right)^2\cos\frac{m\pi}{T}t\right]_{t=0}^{T}$$

When m is an odd number, this gives $a_m = -4VT/Lm^2\pi^2$. When m is even, $a_m = 0$. The desired series is then, by (23),

$$i = \frac{VT}{2L} - \frac{4VT}{L\pi^2}\left(\cos\frac{\pi}{T}t + \frac{1}{9}\cos\frac{3\pi}{T}t + \frac{1}{25}\cos\frac{5\pi}{T}t + \cdots\right)$$

PROBLEMS

1. Write a Fourier sine series representing the current of the preceding example in the interval from $t = 0$ to $t = \pi$.

2. In a certain diode the current is given approximately by $i = kv^2$, where k is a constant and v is the applied voltage. Write a Fourier cosine series approximating the current in the interval from $v = 0$ to $v = \pi$ volts.

3. In the interval from $t = 0$ to $t = T$ a voltage has a constant value of π volts. Express it as a Fourier sine series valid from $t = 0$ to $t = T$.

4. The current in a circuit is $i = \cosh t$. Write a Fourier cosine series describing the current in the interval from $t = 0$ to $t = T$.

5. Express the current of Prob. 4 as a Fourier sine series valid from $t = 0$ to $t = T$.

6. A current $i = 2 \sin \omega t$ flows in a resistor of 10 ohms. Find a Fourier cosine series for the instantaneous power p from $\omega t = 0$ to $\omega t = T$.

REFERENCES

1. E. Goursat: "A Course in Mathematical Analysis," trans. E. R. Hedrick, 2d ed., vol. I, pp. 411–421, Ginn & Company, Boston, 1904.
2. T. J. I'A. Bromwich: "An Introduction to the Theory of Infinite Series," pp. 115*ff*., The Macmillan Company, London, 1908.

Part Six

DIFFERENTIAL EQUATIONS

20

Introduction to Differential Equations

In the preceding chapters we have often been given equations which involved differentials or derivatives and we have been required to find the corresponding functions. Thus far these problems have amounted principally to problems in integration. For example, we have found that

$$\text{if} \quad \frac{dy}{dx} = y \quad \text{then} \quad y = Ce^x \tag{1}$$

or

$$\text{if} \quad \frac{dy}{dx} = e^x \quad \text{then} \quad y = e^x + C \tag{2}$$

In this chapter we shall solve relationships of a more general nature, which are important in the study of electricity.

20-1 Definitions. If an equation involves derivatives or differentials, it is a *differential equation*. If the equation includes no partial derivatives, it is an *ordinary differential equation*. Thus, in the strictest sense, even the simple relations (1) and (2) may be thought of as ordinary differential

equations. We shall not consider *partial differential equations* (those including partial derivatives) in this chapter.

➤ The *order* of a differential equation is the order of the highest derivative which is involved.

Thus (1) and (2) are of the *first order*, as is

$$\frac{dy}{dx} - y = x^2 \tag{3}$$

The equation

$$\frac{d^3y}{dx^3} + \frac{dy}{dx} = \cos x \tag{4}$$

is an ordinary differential equation of the third order.

➤ The *degree* of a differential equation is the power to which the highest derivative involved is found to be raised

after the equation has been cleared of any radicals or fractions involving derivatives. Equations (1) to (4) are of the first degree, but the equation

$$\left(\frac{d^3y}{dx^3}\right)^2 - \left(\frac{dy}{dx}\right)^4 - y = 0 \tag{5}$$

is of the third order and *second* degree. In this chapter we shall study only equations of the first degree.

20-2 Solutions

➤ A solution of a differential equation is a relationship between the variables which is free of derivatives or differentials and which satisfies the differential equation.

Example 1. The equation $y = x^3/6 + 12$ is a solution of the differential equation $d^2y/dx^2 = x$. This may be shown by differentiating the first equation twice, obtaining the given differential equation.

Example 2. The relation $y = x^3 + 2x$ is a solution of the differential equation $dy/dx = 3x^2 + 2$. This may be verified by substituting $x^3 + 2x$ for y in the differential equation. This gives the result $1 = 1$, or as we say, reduces the differential equation to an identity, showing that it is satisfied by the solution given.

If we return to the example of Eq. (1), $dy/dx = y$, we note that the solution was obtained by (*a*) rearranging the quantities involved so that the integrations could be performed and then (*b*) carrying out the integrations. The problem of solving a differential equation is essentially one of integration; however, the variables and their differentials may be related in such a manner that we are put to some trouble to get the equation ready for integration.

A solution of a differential equation is often called an *integral* of the equation. In the study of differential equations an ordinary integration (as $\int x\,dx$, or $\int e^x \cos x\,dx$, for example) is commonly called a *quadrature*.

Equations (1) and (2) are first-order equations inasmuch as they have no derivatives higher than the first. Notice that each has *one* arbitrary constant in its solution.

Now consider the second-order equation

$$\frac{d^2y}{dx^2} = \sin x \tag{6}$$

This can be written

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \sin x$$

Temporarily assigning the symbol p for dy/dx,

$$\frac{dp}{dx} = \sin x$$

Multiplying by dx and integrating,

$$p = -\cos x + C \qquad \text{or} \qquad \frac{dy}{dx} = -\cos x + C$$

Again multiplying by dx and integrating,

$$y = -\sin x + Cx + C_1 \tag{7}$$

Here, we observe that (7), our solution for the second-order equation (6), contains *two* arbitrary constants C and C_1. It can be proved that under appropriate conditions an ordinary differential equation of any order n can have a solution containing n arbitrary constants.[1]

As you may have guessed from the previous discussion, a given differential equation of order n can have more than one solution. A solution containing n arbitrary constants is called a *general solution*. Unless otherwise required, we shall consider an equation solved when a general solution has been found.

When specific values are assigned to any of the arbitrary constants in a general solution, we then refer to the solution as a *particular solution* (or a *particular integral*).

The values of the arbitrary constants may often be established by known *boundary conditions*. In physical problems these conditions are most generally *initial values* of the quantities involved in the equation. Consider, for instance, a differential equation which shows how a current

varies with time, starting with $t = 0$. The initial conditions might involve such values as (*a*) the quantities of charges in the various capacitors when $t = 0$, (*b*) the rates of change of current through the various inductors when $t = 0$, and (*c*) the amounts of current flowing in the different resistors when $t = 0$. An even simpler example follows.

Example 3. Consider the equation which states Ohm's law for a simple resistive dc circuit, where a battery of voltage V and a load of resistance R are connected together:

$$I = \frac{V}{R} \tag{8}$$

The current I is the rate at which the charge q is being withdrawn from the battery:

$$\frac{dq}{dt} = \frac{V}{R} \tag{9}$$

Multiplying by dt and integrating,

$$q = \frac{V}{R}t + K \tag{10}$$

which expresses the total charge which has been transferred from the battery after a time t. The constant K is *the amount of charge which had already been transferred* at the time of closing the circuit, when $t = 0$.

Equation (10) is a general solution for (9) since it includes the single arbitrary constant K required for a general solution of a first-order equation. If the battery had been fully charged at the time the circuit was closed, then K would be zero, giving a *particular* solution

$$q = \frac{V}{R}t \tag{11}$$

But if the battery had been discharged by the amount of 10 coulombs prior to closing of the circuit, the *total* amount of charge transferred after any time t would be given by letting $K = 10$. This gives another particular solution of (9):

$$q = \frac{V}{R}t + 10 \tag{12}$$

The simple example above illustrates that the values of the constants in a particular solution of a differential equation representing physical quantities can often be determined from known initial conditions. Note that V and R in this equation are *not* arbitrary constants: they are known quantities, called *parameters*, in the original differential equation. Arbitrary constants arise in the solution of a differential equation, not in the differential equation itself.

QUESTIONS

1. What is a differential equation?
2. What is an ordinary differential equation? A partial differential equation?
3. What is the *order* of a differential equation? The *degree* of a differential equation?
4. What is a *solution* or *integral* of a differential equation?
5. In the study of differential equations, what is the meaning of the term *quadrature*?
6. What is a general solution of a differential equation?
7. What is a *particular solution* or *particular integral* of a differential equation?

PROBLEMS

In Probs. 1 to 5 give the order of the equations.

1. $\left(\dfrac{d^2y}{dx^2}\right)^2 + \left(\dfrac{dy}{dx}\right)^5 - y = 0$

2. $\left(\dfrac{d^3y}{dx^3}\right)^3 - \dfrac{dy}{dx} - 2x = 0$

3. $L\dfrac{di}{dt} + Ri = V$

4. $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = 100 \sin 377t$

5. $\dfrac{d^2y}{dx^2} - \left(1 - \dfrac{dy}{dx}\right)^{3/2} = 0$

6 to 10. In Probs. 1 to 5 state the degree of the equation.

11 to 15. In Probs. 1 to 5 state the greatest number of essential arbitrary constants which might be contained in a solution.

20-3 Equations with variables separable. We now take up the solving of differential equations. For brevity, we shall limit our work to a few important kinds of equations.

Example 1. Solve the equation

$$\frac{1}{2x}\frac{dy}{dx} - y = 0 \tag{13}$$

It will be noticed that if we multiply each term by $2x\,dx$ and then divide each term by y, the equation becomes

$$\frac{dy}{y} - 2x\,dx = 0 \tag{14}$$

Carrying out the quadratures, we get the solution

$$\ln y - x^2 + C = 0$$

To simplify this integral, write the constant in the form $\ln K$:

$$\ln y - x^2 + \ln K = 0$$

Taking antilogarithms,

$$Ky = e^{x^2} \tag{15}$$

a general solution of (13).

A differential equation is said to have its *variables separable* if it is of the form

$$M\,dx + N\,dy = 0 \tag{16}$$

where M is a function of x alone or a constant and N is a function of y alone or a constant. Equation (13) takes this form when written in the manner (14).

Example 2. Assume a series RL circuit with a constant emf V applied. Here we have

$$L\frac{di}{dt} + Ri = V \tag{17}$$

The solution is accomplished as follows:

$$\frac{di}{dt} = \frac{V - Ri}{L} \qquad \text{or} \qquad di = \frac{1}{L}(V - Ri)\,dt$$

Multiplying and dividing the right member by R,

$$di = \frac{R}{L}\left(\frac{V}{R} - i\right)dt \qquad \text{or} \qquad \frac{di}{V/R - i} = \frac{R}{L}\,dt$$

The variables are now separated, and the quadratures can be found:

$$-\ln\left(\frac{V}{R} - i\right) = \frac{R}{L}t - \ln K$$

Multiplying each term by -1 and taking antilogarithms,

$$\frac{V}{R} - i = Ke^{-Rt/L} \qquad \text{or} \qquad i = \frac{V}{R} - Ke^{-Rt/L} \tag{18}$$

This is a general solution for (17). If there is no initial current, we may substitute the initial conditions $i = 0$, $t = 0$ into (18), finding $K = V/R$. The particular solution corresponding to this condition is

$$i = \frac{V}{R}(1 - e^{-Rt/L}) \tag{19}$$

Figure 20-1 shows the current function (19).

In Example 2, the general solution (18) contains two terms, one being V/R. The other is $-Ke^{-Rt/L}$, an exponential which takes decreasing values as time progresses. The first term is the amount of current which would flow in the absence of inductance and is the value approached by the current in the actual circuit as time goes on. This first term is

called the *steady-state solution* of (17). The term $-Ke^{-Rt/L}$ represents a deviation from the "final" value approached by the current. This term is called the *transient solution* of (17). These relationships are shown in Fig. 20-2.

Differential equations permit us to treat both the transient and the steady-state conditions in a circuit or other physical system. A student

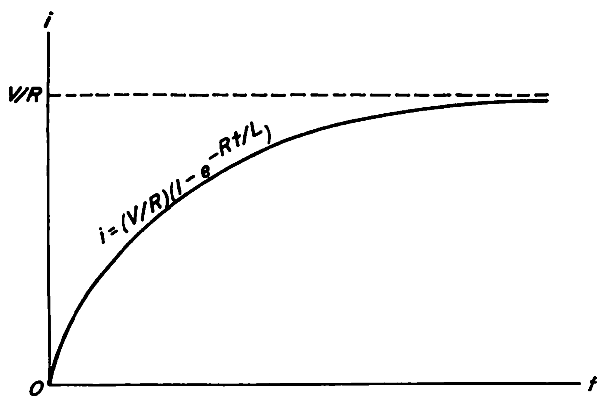


Fig. 20-1

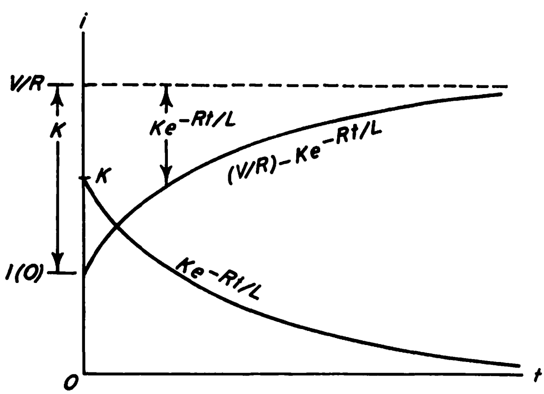


Fig. 20-2

equipped only with algebra and trigonometry would be forced to conclude that the current is simply $i = V/R$, the obvious Ohm's-law solution.

We may describe a *transient* wave as a wave existing while conditions are changing from one steady state to another. In Example 2, the original steady state was that in which the current was zero. The new steady state involves a current $i = V/R$. The term $-Ke^{-Rt/L}$ provides a transition from one state to the other.

The transient response of equipment is often very important. The pulse techniques of radar, loran, television, and related systems require

strict attention to the transient behavior of circuits. The transient response of even a system which is not basically a pulse system is often important. As an example, a sound system may have satisfactory steady-state responses, yet suddenly applied signals common in speech and music could produce unpleasant transient waves.

PROBLEMS

Solve the equations in Probs. 1 to 5.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$
2. $x\dfrac{dy}{dx} + y = 0$
3. $dy = e^{x+y}\, dx$
4. $e^x\, dy - \cos y\, dx = 0$
5. $(y + 2)\, dx + (x - 1)\, dy = 0$

6. An originally discharged capacitance C is charged through a resistance R from a source of constant emf V. Find a formula for the charge q in the capacitor at any instant.

7. In an experiment, charged ions recombined at a rate $dN/dt = -aN^2$, where a was a constant. Get a formula for N.

8. A charged particle of mass m is acted upon by a steady force $\mathbf{F}$. If the particle encounters a resistance proportional to its speed, the differential equation of its motion is $m\, d\mathbf{v}/dt + k\mathbf{v} = \mathbf{F}$. Find an equation for $\mathbf{v}$.

9. The oil dashpot in a time-delay relay has a cross-sectional area A and an opening of area a. The piston displacement s varies with time in such a way that $A\, ds/dt = -Kas$. Find a formula for s in terms of t.

10. A resistance R is connected in parallel with a capacitance C. If, over a certain interval, a steady current I is sent through the combination, get an equation for the voltage drop v across the combination. Assume the capacitor to be initially discharged.

11. An inductor of negligible resistance and of inductance L is connected in parallel with a resistor of resistance R. A constant-current generator supplies a current I to the parallel combination. Find a formula for the voltage drop v across the combination, assuming that prior to the application of I there were no voltages or currents acting in the circuit.

12. A charged particle of weight $\mathbf{W}$ falls from rest against an upward force $\mathbf{F}$ exerted by an electric field. Assuming that the resistance offered by the air is proportional to the speed of the particle, what equation gives the speed $\mathbf{v}$ of the particle?

13. If the particle of Prob. 8 encounters a resistance proportional to its speed squared, get a formula for $\mathbf{v}$.

20-4 Linear equations of first order. If in a differential equation, the dependent variable and its derivatives appear in no powers other than the first, the equation is said to be *linear*. The standard form for the linear equation of the first order is

$$\frac{dy}{dx} + Py = Q \qquad (20)$$

where P and Q are functions of x or constants.

To get a method for solving such equations, first multiply the equation by some suitable *integrating factor* S, which is a function of x as yet undetermined:

$$S\,dy + yPS\,dx = SQ\,dx \tag{21}$$

If we can find a value for S such that

$$dS = PS\,dx \tag{22}$$

then (21) will become

$$S\,dy + y\,dS = SQ\,dx \tag{23}$$

Assume for the moment that we shall be able to find such a value for S. Turning to (23), we recall from our work in differentiation that the differential of a product is $d(uv) = v\,du + u\,dv$. Thus (23) becomes

$$d(Sy) = SQ\,dx \qquad \text{or} \qquad Sy = \int SQ\,dx + C \tag{24}$$

Returning to (22), which is a separable equation, we solve:

$$\frac{dS}{S} = P\,dx$$

from which

$$\ln S = \int P\,dx \qquad \text{or} \qquad S = e^{\int P\,dx} \tag{25}$$

Now we have two formulas, (24) and (25), into which it is only necessary to substitute given values in order to solve an equation of form (20). If

$$\frac{dy}{dx} + Py = Q \tag{20}$$

then

$$Sy = \int SQ\,dx + C \tag{24}$$

where

$$S = e^{\int P\,dx} \tag{25}$$

[In (25) the constant of integration is made zero to get the simplest form for S.]

20-5 Example of a linear first-order equation. In a series RL circuit, in which the values of inductance and resistance are constant, we have

$$L\frac{di}{dt} + Ri = v \tag{26}$$

where v is the applied emf as a function of time. This has already been solved as a separable equation, in Sec. 20-3, for the special case in which

$v = V$, a constant. Now we get the solution for cases where v varies with time. Dividing by L,

$$\frac{di}{dt} + \frac{R}{L} i = \frac{1}{L} v$$

which is of the form (20). The integrating factor S is, by (25),

$$S = \exp \int \frac{R}{L} dt = e^{Rt/L}$$

By (24),

$$e^{Rt/L} i = \frac{1}{L} \int e^{Rt/L} v \, dt + K$$

or

$$i = \frac{1}{L} e^{-Rt/L} \int e^{Rt/L} v \, dt + K e^{-Rt/L} \tag{27}$$

The latter expression is a general solution of (26). As soon as the form of v is specified, we can evaluate the quadrature $\int e^{Rt/L} v \, dt$. If, for example, a sine wave of voltage is applied to the circuit, we may write $v = V \sin \omega t$. Making this substitution into (27) and carrying out the integration, we have

$$i = \frac{V}{R^2 + L^2\omega^2} (R \sin \omega t - L\omega \cos \omega t) + K e^{-Rt/L} \tag{28}$$

In (28), the first term in the right member is the steady-state current. It indicates the form which the current function approaches as the circuit is left closed for a longer and longer time. The remaining form in the right member, containing the arbitrary constant K, is the transient term. Since the exponent is negative, this term decreases as time goes on. Theoretically, the circuit never reaches the steady-state condition but only approaches it. Actually, the transient term usually becomes immeasurably small in a short time.

If in (28) we let $i = 0$ when $t = 0$, we get the current equation for the case when the circuit is closed *precisely* at the zero point which begins a voltage cycle:

$$i = \frac{V}{R^2 + L^2\omega^2} (R \sin \omega t - L\omega \cos \omega t) + \frac{VL\omega}{R^2 + L^2\omega^2} e^{-Rt/L} \tag{29}$$

Figure 20-3 shows an application of this result. At time $t = 0$ a short circuit occurs across a transformer secondary winding whose resistance is small compared with its inductance. During the first few succeeding cycles a circuit breaker in the transformer secondary circuit would have to carry current peaks much greater than the later steady-state short-

circuit current. This condition results from the unsymmetrical or so-called "dc" component represented by the exponential term in (29).

The exact form taken by the starting current depends upon the point on the voltage cycle at which the circuit is closed and upon the ratio of the inductance to the resistance of the circuit.

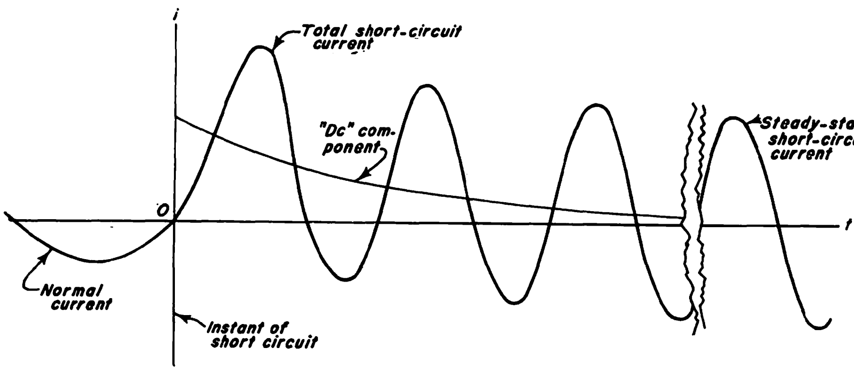


Fig. 20-3

Sometimes it is handy to write (29) in a form involving only one trigonometric function. Consider an angle θ such that its cosine is proportional to R and its sine is proportional to $L\omega$:

$$R = k \cos \theta \qquad \text{and} \qquad L\omega = k \sin \theta$$

To find the value of k, we square the above equations:

$$R^2 = k^2 \cos^2 \theta \qquad \text{and} \qquad L^2\omega^2 = k^2 \sin^2 \theta$$

Adding the two latter equations,

$$R^2 + L^2\omega^2 = k^2(\sin^2 \theta + \cos^2 \theta)$$

Since the quantity in parentheses is equal to 1, we have

$$k = (R^2 + L^2\omega^2)^{1/2}$$

which makes

$$R = (R^2 + L^2\omega^2)^{1/2} \cos \theta \qquad \text{and} \qquad L\omega = (R^2 + L^2\omega^2)^{1/2} \sin \theta$$

Inserting these forms into (29),

$$i = \frac{V}{\sqrt{R^2 + L^2\omega^2}} (\sin \omega t \cos \theta - \sin \theta \cos \omega t) + \frac{VL\omega}{R^2 + L^2\omega^2} e^{-Rt/L} \tag{29a}$$

Applying the trigonometric identity

$$\sin x \cos y = \frac{\sin (x + y) + \sin (x - y)}{2}$$

and using the fact that $\sin(-x) = -\sin x$, we convert (29*a*) to

$$i = \frac{V}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega t - \theta) + \frac{VL\omega}{R^2 + L^2\omega^2} e^{-Rt/L} \qquad (29b)$$

Here $(R^2 + L^2\omega^2)^{1/2}$ is referred to as the *impedance* Z of the circuit, and θ is the *phase angle* of that impedance. Thus the steady-state solution, involving the first term only of the right member, takes the familiar form $i = (V/Z)\sin(\omega t - \theta)$.

Considerations in the preceding example contain the answers to some of the questions brought up by beginning students in electricity, who may ask, "How, in a series *RL* circuit of small *R*, does the current *get into* a phase position of lagging an applied ac voltage by about 90°?" The answers to this question, and to a corresponding one about a series *RC* circuit, are contained in the transient terms of the solutions of appropriate differential equations.

QUESTIONS

1. A knowledge of calculus is particularly helpful in studying which types of waveforms—steady-state or transient?
2. What is a linear differential equation?
3. State a procedure for solving equations of the form $dy/dx + Py = Q$, where P and Q are functions of x or constants.

PROBLEMS

Obtain solutions for Probs. 1 to 4.

1. $\dfrac{dy}{dx} + 2xy = \exp(-x^2)$

2. $\dfrac{dy}{dx} = e^x + y$

3. $\dfrac{dy}{dx} = \dfrac{1 + xy}{1 - x^2}$

4. $(y \cot x - \csc x)\, dx + dy = 0$

5. Derive an equation for the current in a series *RL* circuit subjected to an emf $v = t$ volts. Let $i = 0$ when $t = 0$.

6. Same as Prob. 5, but let $v = e^{-Rt/L}$ and $i = 1$ when $t = 0$.

7. When a rotating armature having a moment of inertia I is given an angular acceleration $d\omega/dt$ against a frictional resistance R, the differential equation of the rotation is $I\, d\omega/dt + R\omega = \mathrm{T}$, where T is the torque. Get a formula for ω if $T = t^2$.

8. An initially discharged capacitance C is connected through a resistance R to a source of emf $v = V \sin \omega t$. Then $R\, di/dt + (1/C)i = dv/dt$. Find an equation for i.

9. The resistance R in a series *RL* circuit is varied so that $R = t$ ohms. If the applied emf is $v = t$ volts, find an equation for the current i. Assume $i = 0$ when $t = 0$.

10. A coil has inductance L and negligible resistance. It is connected in parallel with a resistance R, and a total current $i = I \sin \omega t$ is sent through the combination. If the circuit was at rest before the current was applied, find an equation for the volt-

age drop v across the combination at any time t. (HINT: After setting up the Kirchhoff's-law equation for the currents, differentiate each term of this equation and solve the resulting differential equation.)

11. A parallel RC circuit was connected to a nonlinear current source which supplied a current varying with both time and voltage according to $i = v^2t$. Find an equation for the voltage drop v across the RC combination. [HINT: Equations of the form $dy/dx + Py = Qy^n$ are called *Bernoulli* equations. (P and Q are assumed to be functions of x.) Such equations are solved as follows: (1) Multiply each term by y^{-n}. (2) Let $u = y^{1-n}$. (3) Solve for u according to the method for linear equations. (4) Substitute y^{1-n} for u in the result.]

12. The armature of a shunt dc generator is taken to have negligible resistance. The field has resistance R and inductance L. When the field circuit is open, an emf V volts is generated as a result of residual magnetism. Assume that, over the initial period of voltage build-up after the field circuit is closed, the *additional* generated voltage is proportional to the field current i. (*a*) Get an equation for the field current during this period. (*b*) Get an equation for the generated voltage during this interval.

20-6 Homogeneous linear differential equations with constant coefficients. We now study equations having the following characteristics:

1. The equations are of higher order than the first; that is, they contain higher derivatives than the first.
2. The equations are linear; that is, they contain no powers other than the first of the dependent variable and its derivatives.
3. The coefficients of the dependent variable and of its derivatives are constants.
4. The right members of the equations are zero.

Equations having these characteristics are called homogeneous linear differential equations of higher order. Such equations are represented by the type form

$$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0 \tag{30}$$

A solution of (30) is useful in itself, since such equations arise in practical situations. In addition, this solution helps us in later courses to solve equations where the right member is not zero. As a specific example, let us solve

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 0 \tag{31}$$

[In actually solving equations it will not be necessary to go through the steps which we shall now use in *developing* a method for solving equations like (31).]

The problem can be handled in various ways. The method we shall

follow begins by considering an equation similar to (31), but of the *first* order:

$$\frac{dy}{dx} + Py = 0$$

This equation readily yields the solution

$$y = Ce^{-Px}$$

We might wonder whether or not an exponential relation of similar form

$$y = Ce^{mx} \tag{32}$$

could be a solution of (31). (We shall see that such is actually the case.) Note that C is an arbitrary constant, while m is a constant which must be evaluated.

Differentiating (32) successively, we get

$$\begin{aligned} \frac{dy}{dx} &= Ce^{mx}m \\ \frac{d^2y}{dx^2} &= Ce^{mx}m^2 \\ \frac{d^3y}{dx^3} &= Ce^{mx}m^3 \end{aligned} \tag{33}$$

If the values (32) and (33) are substituted in (31), the latter becomes

$$(m^3 + 3m^2 - m - 3)Ce^{mx} = 0 \tag{34}$$

If (34) is true, then by definition $y = Ce^{mx}$ is a solution of (31). But (34) is correct if suitable values are assigned to m. The problem is one of solving

$$m^3 + 3m^2 - m - 3 = 0 \tag{35}$$

From algebra, we know that there are three values of m (some of them perhaps complex and not all necessarily distinct) satisfying a third-degree equation such as (35). In this instance we can find them by inspection:

$$m = 1, -1, \text{ or } -3 \tag{36}$$

Substituting (36) in (32), we find these solutions for (31):

$$y = C_1e^x \qquad y = C_2e^{-x} \qquad \text{and} \qquad y = C_3e^{-3x} \tag{37}$$

This means that if any one of the values in (37) were substituted into the given equation (31), the equation would be satisfied. Moreover, if you should go to the trouble of substituting the *sum* of these solutions, that is

$$y = C_1e^x + C_2e^{-x} + C_3e^{-3x} \tag{38}$$

into (31), you would find that this, too, is a solution. (This result springs from the fact that the derivative of a sum of terms is equal to the sum of the derivatives of the terms.) Thus Eq. (38) is a *general* solution to

(31), since it includes the three arbitrary constants required in a general solution of a third-order differential equation. And (37) provides three *particular* solutions to (31), since each of these solutions is of the form (38) with some of the C's having the particular value zero.

Our procedure for solving equations of the form (30) then boils down to simply the following:

1. Write the *auxiliary equation* (*characteristic equation*), corresponding to (30):

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1} + a_n = 0 \tag{39}$$

2. Solve (39) for m, obtaining m_1, m_2, etc., as roots.
3. Write the solution

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x} \tag{40}$$

20-7 Repeated roots. In some equations the auxiliary relation (39) will have two or more *equal* roots (as well as perhaps some other single roots). For example, suppose that (39) yielded

$$m = 1, 3, 3, \text{ and } 3$$

It is true that a solution of the form

$$y = C_1 e^x + C_2 e^{3x} + C_3 e^{3x} + C_4 e^{3x} \tag{40a}$$

would satisfy the original differential equation. But this cannot be a *general* solution, for (40a) can also be written $y = C_1 e^x + (C_2 + C_3 + C_4)e^{3x}$, in which the sum within the parentheses can be replaced by a single constant C_5:

$$y = C_1 e^x + C_5 e^{3x} \tag{40b}$$

In this form, there are only *two* arbitrary constants.

Texts on differential equations show that a general solution is had by multiplying the repeating terms of (40a) by successively higher powers of x, getting

$$y = C_1 e^x + C_2 e^{3x} + C_3 x e^{3x} + C_4 x^2 e^{3x} \tag{40c}$$

20-8 Complex roots. If the coefficients a_1, a_2, etc., in (39) are real, as is usually the case in practical problems, then any imaginary or complex roots m_1, m_2, etc., of the auxiliary equation must appear in conjugate pairs. (This result appears in texts on the theory of equations.) Thus, if a root m_1 is equal to $a + jb$, there must exist a second root $m_2 = a - jb$. Terms associated with these roots will appear in a general solution of the differential equation, as follows:

$$C_1 e^{(a+jb)x} + C_2 e^{(a-jb)x} \qquad \text{or} \qquad e^{ax}(C_1 e^{jbx} + C_2 e^{-jbx})$$

By (27) and (30) of Chap. 17, this is

$$e^{ax}[(C_1 + C_2) \cos bx + j(C_1 - C_2) \sin bx]$$

Let $C_1 + C_2$ be signified by A, and let $j(C_1 - C_2)$ be indicated by B. The above form becomes

$$e^{ax}(A \cos bx + B \sin bx) \tag{41}$$

so that, when a pair of complex roots appears in the solution of the auxiliary equation, we may at once write the corresponding part of the solution of the differential equation in the form (41). Here A and B are the arbitrary constants.

Other forms of (41) can be derived. Multiplying and dividing (41) by $(A^2 + B^2)^{1/2}$, we get

$$e^{ax}\sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \cos bx + \frac{B}{\sqrt{A^2 + B^2}} \sin bx\right) \tag{42}$$

Consider an auxiliary angle θ, such that $\sin \theta = A/(A^2 + B^2)^{1/2}$ and such that $\cos \theta = B/(A^2 + B^2)^{1/2}$. This makes (42)

$$e^{ax}\sqrt{A^2 + B^2}\,(\sin \theta \cos bx + \cos \theta \sin bx)$$

Let the radical expression in this form be represented by C. Trigonometric identities make the entire form read

$$Ce^{ax}[\tfrac{1}{2} \sin (bx + \theta) + \tfrac{1}{2} \sin (bx - \theta) + \tfrac{1}{2} \sin (bx + \theta) + \tfrac{1}{2} \sin (\theta - bx)] \tag{43}$$

The last term can be written $-\tfrac{1}{2} \sin (bx - \theta)$, so that (43) reduces to

$$Ce^{ax} \sin (bx + \theta) \tag{44}$$

where the arbitrary constants are C and θ.

If in expression (42) we had let the auxiliary angle be ϕ, such that $\sin \phi = -B/(A^2 + B^2)^{1/2}$ and $\cos \phi = A/(A^2 + B^2)^{1/2}$, then a calculation much like that above would have given us

$$Ce^{ax} \cos (bx + \phi) \tag{45}$$

where C and ϕ are arbitrary constants.

20-9 Transients in series *RLC* circuits. *a. Solving the equation.* In the circuit of Fig. 20-4, consider the capacitor to be originally discharged and the switch to be closed at a time $t = 0$. The voltages around the circuit are related by the equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = V \tag{46}$$

Differentiating and dividing each term by L,

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \tag{47}$$

The solution of this equation will have the form

$$i = K_1e^{m_1t} + K_2e^{m_2t} \tag{48}$$

First we find m_1 and m_2. The auxiliary equation is

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

By the quadratic formula,

$$m_1 = \frac{-R + \sqrt{R^2 - 4L/C}}{2L} \quad \text{and} \quad m_2 = \frac{-R - \sqrt{R^2 - 4L/C}}{2L} \tag{49}$$

We now evaluate the arbitrary constants K_1 and K_2. This is done through a knowledge of the circuit conditions when $t = 0$. Since the

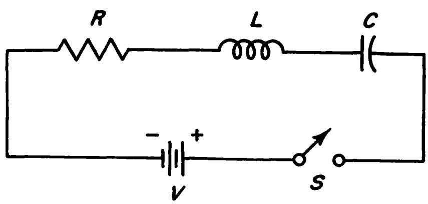


Fig. 20-4

current in the inductor and the charge in the capacitor cannot rise instantly from zero to some other value, we have

$$i = 0 \quad \text{when} \quad t = 0 \tag{50}$$

and

$$\int i\,dt = 0 \quad \text{when} \quad t = 0 \tag{51}$$

These values substituted into (46) give a further initial condition:

$$\frac{di}{dt} = \frac{V}{L} \quad \text{when } t = 0 \tag{52}$$

Condition (50) gives us from (48):

$$K_2 = -K_1 \tag{53}$$

Substituting (53) in (48), and differentiating,

$$i = K_1(e^{m_1t} - e^{m_2t})$$

$$\frac{di}{dt} = K_1(m_1e^{m_1t} - m_2e^{m_2t}) \tag{54}$$

Letting $t = 0$ in (54) and equating the result to (52), we solve for K_1:

$$K_1 = \frac{V}{\sqrt{R^2 - 4L/C}} \tag{55}$$

Substituting (49), (54), and (55) into (48),

$$i = \frac{V}{\sqrt{R^2 + 4L/C}}\left[\exp\left(\frac{-R + \sqrt{R^2 - 4L/C}}{2L}t\right) - \exp\left(\frac{-R - \sqrt{R^2 - 4L/C}}{2L}t\right)\right] \tag{56}$$

or

$$i = \frac{Ve^{-Rt/2L}}{\sqrt{R^2 + 4L/C}}\left[\exp\left(\frac{t}{2L}\sqrt{R^2 - \frac{4L}{C}}\right) - \exp\left(\frac{-t}{2L}\sqrt{R^2 - \frac{4L}{C}}\right)\right] \tag{57}$$

b. Case I. *Real roots.* When the components of the circuit have values such that

$$R^2 > \frac{4L}{C} \tag{58}$$

the solutions of (49) will be real. Upon the closing of the switch in such a circuit, there will be an inrush of current which will at first rise, then decrease as the capacitor voltage approaches the battery voltage. It is convenient in this case to write (57) in the form

$$i = \frac{2Ve^{-Rt/2L}}{\sqrt{R^2 - 4L/C}}\sinh\left(\frac{t}{2L}\sqrt{R^2 - \frac{4L}{C}}\right) \tag{59}$$

A graph of such a current is shown in Fig. 20-5. The current, after once beginning to decrease, never again increases but approaches zero.

c. Case II. *Damped oscillations.* If the series *RLC* circuit has components such that

$$R^2 < \frac{4L}{C} \tag{60}$$

we may benefit by writing the radical expression of (57) as

$$\sqrt{-\left(\frac{4L}{C} - R^2\right)} \qquad \text{or} \qquad j\sqrt{\frac{4L}{C} - R^2} \tag{61}$$

Substituting the latter form into (59) and applying the identity

$$\sinh jx = j \sin x$$

(Appendix, Table 5), we make (59) read

$$i = \frac{2Ve^{-Rt/2L}}{\sqrt{4L/C - R^2}}\sin\left(\frac{t}{2L}\sqrt{\frac{4L}{C} - R^2}\right) \tag{62}$$

This expression is the product of three factors—a constant, a decreasing exponential called the *damping factor*, and a real sinusoidal oscillation. The result is a current function which approaches zero in an oscillating manner as shown in Fig. 20-6. This result is called a *damped oscillation.*

It is of importance in describing the behavior of circuits when they are subjected to a suddenly applied voltage (*shock excitation*).

d. CASE III. *Critical damping.* Between the two foregoing conditions there is a somewhat hypothetical state which applies when

$$R^2 = \frac{4L}{C} \tag{63}$$

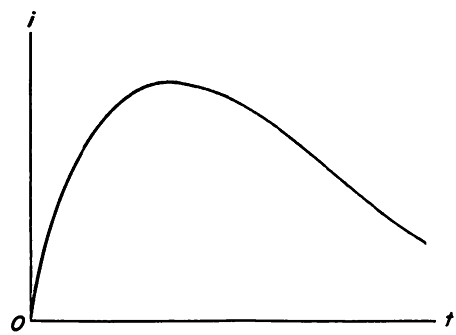


Fig. 20-5

This condition indicates that an arbitrarily small change (in the proper direction) of the circuit values will make the circuit subject to damped oscillations. This borderline condition is called *critical damping*. A circuit which is subject to damped oscillations is said to have *less than critical damping* (as when (59) applies). If damped oscillations cannot occur, as when (60) applies, the damping is *greater than critical.*

The idea of *damping* is important in mechanical applications as well as in electrical circuits. Assume that a galvanometer has an equation for its pointer displacement similar in form to (59), with friction, mass, and stiffness substituted for the analogous electrical quantities of resist-

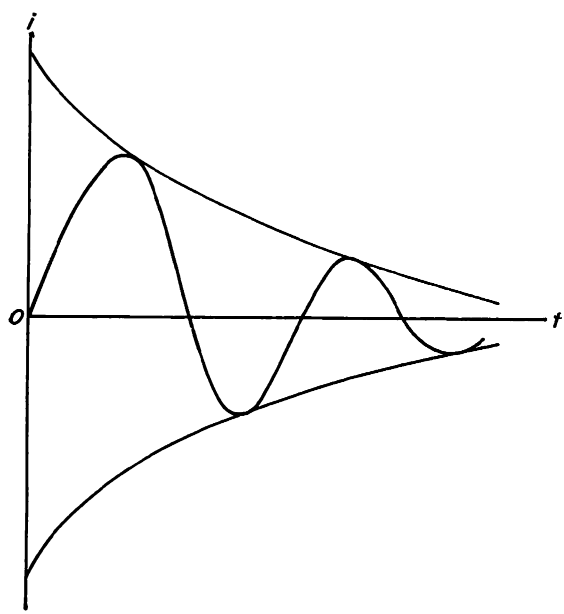


Fig. 20-6

ance, inductance, and the reciprocal of capacitance. (The constants of the electric circuit must also be considered in setting up the equation.) If the damping is critical or greater, the pointer will rise to its final position without oscillating about the final reading. The most rapid rise without oscillation is obtained with exactly critical damping.

QUESTIONS

1. Describe a homogeneous linear differential equation of higher order than the first and with constant coefficients.
2. What procedure is used in solving homogeneous linear differential equations of higher order having constant coefficients?
3. If the auxiliary equation associated with a homogeneous linear differential equation having constant coefficients has repeated roots, what procedure is applied to the repeating terms in the solution of the differential equation?
4. The auxiliary equation associated with a certain homogeneous linear differential equation having constant coefficients has only a pair of complex conjugate roots. Give three forms of a general solution of this differential equation.
5. What condition must apply in order for damped oscillations to occur when a series RLC circuit is connected to a dc source?
6. What is meant by critical damping?

PROBLEMS

Obtain general solutions for the equations of Probs. 1 to 8.

1. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$

2. $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 25y = 0$

3. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} = 0$

4. $\dfrac{d^2y}{dx^2} - K^2y = 0$

5. $\dfrac{d^3y}{dx^3} - 5\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} - 4 = 0$

6. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

7. $\dfrac{d^2y}{dx^2} + 4y = 0$

8. $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 5\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4 = 0$

9. A dc signaling voltage is applied to a transmission line. The voltage v at a point on the line and the distance s of the point from the end of the line are related by $d^2v/ds^2 - \alpha^2 v = 0$, where α is the *attenuation constant* of the line. (*a*) Find a formula, in exponential form, for v. (*b*) Express v in hyperbolic form.

10. The capacitor in a series RLC circuit is charged to a voltage V. If the circuit is closed upon itself at $t = 0$, with no external voltage source, find an expression for i. Assume damping less than critical. [NOTE: The result is similar to that of Sec. 20-9(*a*), except for the boundary conditions.]

11. A mechanism in a semiautomatic telegraph key includes a vibrating body whose mass is M units mounted on a spring whose stiffness is S units. The frictional resistance to the vibration is F units. (*a*) Write a differential equation relating the deflection y of the body and time t. (*b*) Solve for y.

12. The deflection, ϕ radians, of an instrument pointer is given by $A\, d^2\phi/dt^2 + B\, d\phi/dt + C\phi = 0$, where A includes the effects of electrical inductance and of

inertia of the pointer and coil; B expresses the effects of friction and circuit resistance, and C allows for the stiffness of the spring and for any capacitance present. What equation expresses ϕ as a function of t if $d\phi/dt = 0$ and $\phi = 1$ when $t = 0$? Assume critical damping.

13. A charged particle of mass M is repelled from a fixed point by a field which varies so that the force is equal to k times the distance s of the particle from the point. (*a*) Write a differential equation relating s and time t. (*b*) Solve the equation.

20-10 Further considerations. Other useful and interesting differential equations are taken up in courses on that subject. In actually working differential equations in engineering, many laborsaving devices are used. Some examples follow:

1. Sometimes the equation is solved for only a particular solution where conditions do not require a general solution.

2. In carrying out the integrations it is often found that the substitution of exponential forms for trigonometric or hyperbolic functions reduces the labor.

3. It may be found that the first few terms of an *infinite-series* solution give an answer of adequate accuracy. This result can sometimes be had with a reasonable amount of labor, where a *closed* solution in terms of the elementary functions might be difficult or impossible.

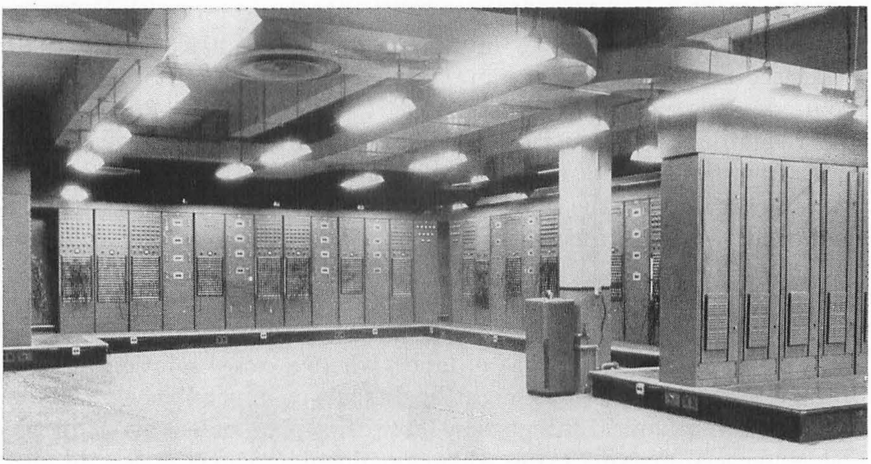
4. While mechanical integration (as with a planimeter) gives, in general, numerical answers rather than functional relations, it is often just such numerical answers that are desired. And the information so obtained is sometimes used to advantage in solving for a functional relationship.[2]

5. Very often there is a mathematical similarity (or *analogy*) between electric-circuit problems and problems in such fields as mechanics and chemistry. It is, then, often practical to obtain numerical solutions of differential equations in these fields by setting up corresponding *electric* circuits and obtaining indications of the results of applying known impulses to these circuits. Equipment for this work is called an *analog computer*,[3] and for many applications, such a computer becomes quite complicated (Fig. 20-7). It has even been possible to make significant studies in the field of economics,[4] where, for instance, electric-circuit elements might represent delay between investment of money and production of goods, time involved in depreciation, distribution of manufactured goods, increase of capital goods, and national consumption. Such studies indicate that economic transients due to the Civil War are still significant in their effect upon our economy.

In circuits having more than one path for the current it is necessary to solve a number of differential equations simultaneously. Besides the traditional methods for such solutions, we have available the interesting *transform* methods by which differential equations are reduced to alge-

braic problems.[5] In fact, it may not even be necessary to set up the differential equations at all.[6]

In this chapter we have assumed in most cases that the circuit elements are *linear* (do not vary with current or voltage). If this assumption is untrue, the solution often requires a high order of skill or may have to be determined experimentally.



Reeves Instrument Corp.

Fig. 20-7

Here, we have limited ourselves to the *analysis* of the action of given circuits. This problem is apart from that of *synthesis*, or the design of circuits intended to respond in a given way to applied impulses. The synthesis problem is generally reserved for advanced courses.

REFERENCES

1. E. L. INCE: "Ordinary Differential Equations," pp. 11–13 and chap. 3, Dover Publications, New York, 1925.
2. T. C. FRY: "Elementary Differential Equations," pp. 69–74, D. Van Nostrand Company, Inc., Princeton, N.J., 1929.
3. F. H. YOUNG: The NOTS REAC, *Am. Math. Monthly*, **60**(4):237–243 (April, 1953).
4. O. J. M. SMITH and H. F. ERDLEY: An Electronic Analogue for an Economic System, *Elec. Eng.*, **71**(4):362–366 (April, 1952).
5. S. GOLDMAN: "Transformation Calculus and Electrical Transients," Prentice-Hall, Inc., Englewood Cliffs, N.J., 1949.
6. G. E. VALLEY, JR. and HENRY WALLMAN: "Vacuum Tube Amplifiers," chap. 1, McGraw-Hill Book Company, Inc., New York, 1948.

Appendix

Table 1 Trigonometric Identities

1. $\sin\theta = \dfrac{1}{\csc\theta}$

2. $\cos\theta = \dfrac{1}{\sec\theta}$

3. $\tan\theta = \dfrac{1}{\cot\theta} = \dfrac{\sin\theta}{\cos\theta}$

4. $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

5. $\sec\theta = \dfrac{1}{\cos\theta}$

6. $\csc\theta = \dfrac{1}{\sin\theta}$

7. $\sin^2\theta + \cos^2\theta = 1$

8. $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

9. $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

10. $\sec^2\theta = 1 + \tan^2\theta$

11. $\csc^2\theta = 1 + \cot^2\theta$

12. $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

13. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

14. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

15. $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

16. $\sin\alpha - \sin\beta = 2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$

17. $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

18. $\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$

19. $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

20. $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

21. $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

22. $\cos\theta + j\sin\theta = e^{j\theta}$

23. $\cos\theta - j\sin\theta = e^{-j\theta}$

Table 2 Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
1.0	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
5.0	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

Table 2 Common Logarithms (Continued)

N	0	1	2	3	4	5	6	7	8	9
5.5	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
5.6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
5.7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
5.8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
5.9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
6.0	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
6.5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
6.8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
7.0	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
7.1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
7.2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
7.3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
7.4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
7.5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
7.6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
7.7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
7.8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
7.9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
8.0	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
8.4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
8.5	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
8.6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
8.7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
8.8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
8.9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
9.0	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
9.1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
9.2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
9.3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
9.4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
9.5	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
9.6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
9.7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
9.8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
9.9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Table 3 Natural Logarithms

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0 000	100	198	296	392	488	583	677	770	862
1.1	953	*044	*133	*222	*310	*398	*484	*570	*655	*740
1.2	0.1 823	906	989	*070	*151	*231	*311	*390	*469	*546
1.3	0.2 624	700	776	852	927	*001	*075	*148	*221	*293
1.4	0.3 365	436	507	577	646	716	784	853	920	988
1.5	0.4 055	121	187	253	318	383	447	511	574	637
1.6	700	762	824	886	947	*008	*068	*128	*188	*247
1.7	0.5 306	365	423	481	539	596	653	710	766	822
1.8	878	933	988	*043	*098	*152	*206	*259	*313	*366
1.9	0.6 419	471	523	575	627	678	729	780	831	881
2.0	931	981	*031	*080	*129	*178	*227	*275	*324	*372
2.1	0.7 419	467	514	561	608	655	701	747	793	839
2.2	885	930	975	*020	*065	*109	*154	*198	*242	*286
2.3	0.8 329	372	416	459	502	544	587	629	671	713
2.4	755	796	838	879	920	961	*002	*042	*083	*123
2.5	0.9 163	203	243	282	322	361	400	439	478	517
2.6	555	594	632	670	708	746	783	821	858	895
2.7	933	969	*006	*043	*080	*116	*152	*188	*225	*260
2.8	1.0 296	332	367	403	438	473	508	543	578	613
2.9	647	682	716	750	784	818	852	886	919	953
3.0	986	*019	*053	*086	*119	*151	*184	*217	*249	*282
3.1	1.1 314	346	378	410	442	474	506	537	569	600
3.2	632	663	694	725	756	787	817	848	878	909
3.3	939	969	*000	*030	*060	*090	*119	*149	*179	*208
3.4	1.2 238	267	296	326	355	384	413	442	470	499
3.5	528	556	585	613	641	669	698	726	754	782
3.6	809	837	865	892	920	947	975	*002	*029	*056
3.7	1.3 083	110	137	164	191	218	244	271	297	324
3.8	350	376	402	429	455	481	507	533	558	584
3.9	610	635	661	686	712	737	762	788	813	838
4.0	863	888	913	938	962	987	*012	*036	*061	*085
4.1	1.4 110	134	159	183	207	231	255	279	303	327
4.2	351	375	398	422	446	469	493	516	540	563
4.3	586	609	633	656	679	702	725	748	770	793
4.4	816	839	861	884	907	929	951	974	996	*019
4.5	1.5 041	063	085	107	129	151	173	195	217	239
4.6	261	282	304	326	347	369	390	412	433	454
4.7	476	497	518	539	560	581	602	623	644	665
4.8	686	707	728	748	769	790	810	831	851	872
4.9	892	913	933	953	974	994	*014	*034	*054	*074
5.0	1.6 094	114	134	154	174	194	214	233	253	273

If given number $n = N \times 10^m$, then $\log_e n = \log_e N + m \log_e 10$. Find $m \log_e 10$ from the following table:

Multiples of $\log_e 10$

$\log_e 10 = 2.3026$	$-\log_e 10 = 7.6974 - 10$
$2 \log_e 10 = 4.6052$	$-2 \log_e 10 = 5.3948 - 10$
$3 \log_e 10 = 6.9078$	$-3 \log_e 10 = 3.0922 - 10$
$4 \log_e 10 = 9.2103$	$-4 \log_e 10 = 0.7897 - 10$
$5 \log_e 10 = 11.5129$	$-5 \log_e 10 = 9.4871 - 20$

Table 3 Natural Logarithms (Continued)

N	0	1	2	3	4	5	6	7	8	9
5.0	1.6 094	114	134	154	174	194	214	233	253	273
5.1	292	312	332	351	371	390	409	429	448	467
5.2	487	506	525	544	563	582	601	620	639	658
5.3	677	696	715	734	752	771	790	808	827	845
5.4	864	882	901	919	938	956	974	993	*011	*029
5.5	1.7 047	066	084	102	120	138	156	174	192	210
5.6	228	246	263	281	299	317	334	352	370	387
5.7	405	422	440	457	475	492	509	527	544	561
5.8	579	596	613	630	647	664	681	699	716	733
5.9	750	766	783	800	817	834	851	867	884	901
6.0	918	934	951	967	984	*001	*017	*034	*050	*066
6.1	1.8 083	099	116	132	148	165	181	197	213	229
6.2	245	262	278	294	310	326	342	358	374	390
6.3	405	421	437	453	469	485	500	516	532	547
6.4	563	579	594	610	625	641	656	672	687	703
6.5	718	733	749	764	779	795	810	825	840	856
6.6	871	886	901	916	931	946	961	976	991	*006
6.7	1.9 021	036	051	066	081	095	110	125	140	155
6.8	169	184	199	213	228	242	257	272	286	301
6.9	315	330	344	359	373	387	402	416	430	445
7.0	459	473	488	502	516	530	544	559	573	587
7.1	601	615	629	643	657	671	685	699	713	727
7.2	741	755	769	782	796	810	824	838	851	865
7.3	879	892	906	920	933	947	961	974	988	*001
7.4	2.0 015	028	042	055	069	082	096	109	122	136
7.5	149	162	176	189	202	215	229	242	255	268
7.6	281	295	308	321	334	347	360	373	386	399
7.7	412	425	438	451	464	477	490	503	516	528
7.8	541	554	567	580	592	605	618	631	643	656
7.9	669	681	694	707	719	732	744	757	769	782
8.0	794	807	819	832	844	857	869	882	894	906
8.1	919	931	943	956	968	980	992	*005	*017	*029
8.2	2.1 041	054	066	080	090	102	114	126	138	150
8.3	163	175	187	199	211	223	235	247	258	270
8.4	282	294	306	318	330	342	353	365	377	389
8.5	401	412	424	436	448	460	471	483	494	506
8.6	518	529	541	552	564	576	587	599	610	622
8.7	633	645	656	668	679	691	702	713	725	736
8.8	748	759	770	782	793	804	815	827	838	849
8.9	861	872	883	894	905	917	928	939	950	961
9.0	972	983	994	*006	*017	*028	*039	*050	*061	*072
9.1	2.2 083	094	105	116	127	137	148	159	170	181
9.2	192	203	214	225	235	246	257	268	279	289
9.3	300	311	322	332	343	354	364	375	386	396
9.4	407	418	428	439	450	460	471	481	492	502
9.5	513	523	534	544	555	565	576	586	597	607
9.6	618	628	638	649	659	670	680	690	701	711
9.7	721	732	742	752	762	773	783	793	803	814
9.8	824	834	844	854	865	875	885	895	905	915
9.9	925	935	946	956	966	976	986	996	*006	*016
10.	2.3 026	036	046	056	066	076	086	096	106	115

Table 4 Exponential Functions

x	e^x	$\log_{10} e^x$	e^{-x}	x	e^x	$\log_{10} e^x$	e^{-x}
0.00	1.0000	0.00000	1.00000	0.50	1.6487	0.21715	0.60653
0.01	1.0101	0.00434	0.99005	0.51	1.6653	0.22149	0.60050
0.02	1.0202	0.00869	0.98020	0.52	1.6820	0.22583	0.59452
0.03	1.0305	0.01303	0.97045	0.53	1.6989	0.23018	0.58860
0.04	1.0408	0.01737	0.96079	0.54	1.7160	0.23452	0.58275
0.05	1.0513	0.02171	0.95123	0.55	1.7333	0.23886	0.57695
0.06	1.0618	0.02606	0.94176	0.56	1.7507	0.24320	0.57121
0.07	1.0725	0.03040	0.93239	0.57	1.7683	0.24755	0.56553
0.08	1.0833	0.03474	0.92312	0.58	1.7860	0.25189	0.55990
0.09	1.0942	0.03909	0.91393	0.59	1.8040	0.25623	0.55433
0.10	1.1052	0.04343	0.90484	0.60	1.8221	0.26058	0.54881
0.11	1.1163	0.04777	0.89583	0.61	1.8404	0.26492	0.54335
0.12	1.1275	0.05212	0.88692	0.62	1.8589	0.26926	0.53794
0.13	1.1388	0.05646	0.87809	0.63	1.8776	0.27361	0.53259
0.14	1.1503	0.06080	0.86936	0.64	1.8965	0.27795	0.52729
0.15	1.1618	0.06514	0.86071	0.65	1.9155	0.28229	0.52205
0.16	1.1735	0.06949	0.85214	0.66	1.9348	0.28663	0.51685
0.17	1.1853	0.07383	0.84366	0.67	1.9542	0.29098	0.51171
0.18	1.1972	0.07817	0.83527	0.68	1.9739	0.29532	0.50662
0.19	1.2092	0.08252	0.82696	0.69	1.9937	0.29966	0.50158
0.20	1.2214	0.08686	0.81873	0.70	2.0138	0.30401	0.49659
0.21	1.2337	0.09120	0.81058	0.71	2.0340	0.30835	0.49164
0.22	1.2461	0.09554	0.80252	0.72	2.0544	0.31269	0.48675
0.23	1.2586	0.09989	0.79453	0.73	2.0751	0.31703	0.48191
0.24	1.2712	0.10423	0.78663	0.74	2.0959	0.32138	0.47711
0.25	1.2840	0.10857	0.77880	0.75	2.1170	0.32572	0.47237
0.26	1.2969	0.11292	0.77105	0.76	2.1383	0.33006	0.46767
0.27	1.3100	0.11726	0.76338	0.77	2.1598	0.33441	0.46301
0.28	1.3231	0.12160	0.75578	0.78	2.1815	0.33875	0.45841
0.29	1.3364	0.12595	0.74826	0.79	2.2034	0.34309	0.45384
0.30	1.3499	0.13029	0.74082	0.80	2.2255	0.34744	0.44933
0.31	1.3634	0.13463	0.73345	0.81	2.2479	0.35178	0.44486
0.32	1.3771	0.13897	0.72615	0.82	2.2705	0.35612	0.44043
0.33	1.3910	0.14332	0.71892	0.83	2.2933	0.36046	0.43605
0.34	1.4049	0.14766	0.71177	0.84	2.3164	0.36481	0.43171
0.35	1.4191	0.15200	0.70469	0.85	2.3396	0.36915	0.42741
0.36	1.4333	0.15635	0.69768	0.86	2.3632	0.37349	0.42316
0.37	1.4477	0.16069	0.69073	0.87	2.3869	0.37784	0.41895
0.38	1.4623	0.16503	0.68386	0.88	2.4109	0.38218	0.41478
0.39	1.4770	0.16937	0.67706	0.89	2.4351	0.38652	0.41066
0.40	1.4918	0.17372	0.67032	0.90	2.4596	0.39087	0.40657
0.41	1.5068	0.17806	0.66365	0.91	2.4843	0.39521	0.40252
0.42	1.5220	0.18240	0.65705	0.92	2.5093	0.39955	0.39852
0.43	1.5373	0.18675	0.65051	0.93	2.5345	0.40389	0.39455
0.44	1.5527	0.19109	0.64404	0.94	2.5600	0.40824	0.39063
0.45	1.5683	0.19543	0.63763	0.95	2.5857	0.41258	0.38674
0.46	1.5841	0.19978	0.63128	0.96	2.6117	0.41692	0.38289
0.47	1.6000	0.20412	0.62500	0.97	2.6379	0.42127	0.37908
0.48	1.6161	0.20846	0.61878	0.98	2.6645	0.42561	0.37531
0.49	1.6323	0.21280	0.61263	0.99	2.6912	0.42995	0.37158
0.50	1.6487	0.21715	0.60653	1.00	2.7183	0.43429	0.36788

Table 4 Exponential Functions (Continued)

x	e^x	$\log_{10} e^x$	e^{-x}	x	e^x	$\log_{10} e^x$	e^{-x}
1.00	2.7183	0.43429	0.36788	1.50	4.4817	0.65144	0.22313
1.01	2.7456	0.43864	0.36422	1.51	4.5267	0.65578	0.22091
1.02	2.7732	0.44298	0.36060	1.52	4.5722	0.66013	0.21871
1.03	2.8011	0.44732	0.35701	1.53	4.6182	0.66447	0.21654
1.04	2.8292	0.45167	0.35345	1.54	4.6646	0.66881	0.21438
1.05	2.8577	0.45601	0.34994	1.55	4.7115	0.67316	0.21225
1.06	2.8864	0.46035	0.34646	1.56	4.7588	0.67750	0.21014
1.07	2.9154	0.46470	0.34301	1.57	4.8066	0.68184	0.20805
1.08	2.9447	0.46904	0.33960	1.58	4.8550	0.68619	0.20598
1.09	2.9743	0.47338	0.33622	1.59	4.9037	0.69053	0.20393
1.10	3.0042	0.47772	0.33287	1.60	4.9530	0.69487	0.20190
1.11	3.0344	0.48207	0.32956	1.61	5.0028	0.69921	0.19989
1.12	3.0649	0.48641	0.32628	1.62	5.0531	0.70356	0.19790
1.13	3.0957	0.49075	0.32303	1.63	5.1039	0.70790	0.19593
1.14	3.1268	0.49510	0.31982	1.64	5.1552	0.71224	0.19398
1.15	3.1582	0.49944	0.31664	1.65	5.2070	0.71659	0.19205
1.16	3.1899	0.50378	0.31349	1.66	5.2593	0.72093	0.19014
1.17	3.2220	0.50812	0.31037	1.67	5.3122	0.72527	0.18825
1.18	3.2544	0.51247	0.30728	1.68	5.3656	0.72961	0 18637
1.19	3.2871	0.51681	0.30422	1.69	5.4195	0.73396	0.18452
1.20	3.3201	0.52115	0.30119	1.70	5.4739	0.73830	0.18268
1.21	3.3535	0.52550	0.29820	1.71	5.5290	0.74264	0.18087
1.22	3.3872	0.52984	0.29523	1.72	5.5845	0.74699	0.17907
1.23	3.4212	0.53418	0.29229	1.73	5.6407	0.75133	0.17728
1.24	3.4556	0.53853	0.28938	1.74	5.6973	0.75567	0.17552
1.25	3.4903	0.54287	0.28650	1.75	5.7546	0.76002	0.17377
1.26	3.5254	0.54721	0.28365	1.76	5.8124	0.76436	0.17204
1.27	3.5609	0.55155	0.28083	1.77	5.8709	0.76870	0.17033
1.28	3.5966	0.55590	0.27804	1.78	5.9299	0.77304	0.16864
1.29	3.6328	0.56024	0.27527	1.79	5.9895	0.77739	0.16696
1.30	3.6693	0.56458	0.27253	1.80	6.0496	0.78173	0.16530
1.31	3.7062	0.56893	0.26982	1.81	6.1104	0.78607	0.16365
1.32	3.7434	0.57327	0.26714	1.82	6.1719	0.79042	0.16203
1.33	3.7810	0.57761	0.26448	1.83	6.2339	0.79476	0.16041
1.34	3.8190	0.58195	0.26185	1.84	6.2965	0.79910	0.15882
1.35	3.8574	0.58630	0.25924	1.85	6.3598	0.80344	0.15724
1.36	3.8962	0.59064	0.25666	1.86	6.4237	0.80779	0.15567
1.37	3.9354	0.59498	0.25411	1.87	6.4883	0.81213	0.15412
1.38	3.9749	0.59933	0.25158	1.88	6.5535	0.81647	0.15259
1.39	4.0149	0.60367	0.24908	1.89	6.6194	0.82082	0.15107
1.40	4.0552	0.60801	0.24660	1.90	6.6859	0.82516	0.14957
1.41	4.0960	0.61236	0.24414	1.91	6.7531	0.82950	0.14808
1.42	4.1371	0.61670	0.24171	1.92	6.8210	0.83385	0.14661
1.43	4.1787	0.62104	0.23931	1.93	6.8895	0.83819	0.14515
1.44	4.2207	0.62538	0.23693	1.94	6.9588	0.84253	0.14370
1 45	4.2631	0.62973	0.23457	1.95	7.0287	0.84687	0.14227
1.46	4.3060	0.63407	0.23224	1.96	7.0993	0.85122	0.14086
1.47	4.3492	0.63841	0.22993	1.97	7.1707	0.85556	0.13946
1.48	4.3929	0.64276	0.22764	1.98	7.2427	0.85990	0.13807
1.49	4.4371	0.64710	0.22537	1.99	7.3155	0.86425	0.13670
1.50	4.4817	0.65144	0.22313	2.00	7.3891	0.86859	0.13534

Table 4 Exponential Functions (Continued)

x	e^x	$\log_{10} e^x$	e^{-x}	x	e^x	$\log_{10} e^x$	e^{-x}
2.00	7.3891	0.86859	0.13534	2.50	12.182	1.08574	0.08208
2.01	7.4633	0.87293	0.13399	2.51	12.305	1.09008	0.08127
2.02	7.5383	0.87727	0.13266	2.52	12.429	1.09442	0.08046
2.03	7.6141	0.88162	0.13134	2.53	12.554	1.09877	0.07966
2.04	7.6906	0.88596	0.13003	2.54	12.680	1.10311	0.07887
2.05	7.7679	0.89030	0.12873	2.55	12.807	1.10745	0.07808
2.06	7.8460	0.89465	0.12745	2.56	12.936	1.11179	0.07730
2.07	7.9248	0.89899	0.12619	2.57	13.066	1.11614	0.07654
2.08	8.0045	0.90333	0.12493	2.58	13.197	1.12048	0.07577
2.09	8.0849	0.90768	0.12369	2.59	13.330	1.12482	0.07502
2.10	8.1662	0.91202	0.12246	2.60	13.464	1.12917	0.07427
2.11	8.2482	0.91636	0.12124	2.61	13.599	1.13351	0.07353
2.12	8.3311	0.92070	0.12003	2.62	13.736	1.13785	0.07280
2.13	8.4149	0.92505	0.11884	2.63	13.874	1.14219	0.07208
2.14	8.4994	0.92939	0.11765	2.64	14.013	1.14654	0.07136
2.15	8.5849	0.93373	0.11648	2.65	14.154	1.15088	0.07065
2.16	8.6711	0.93808	0.11533	2.66	14.296	1.15522	0.06995
2.17	8.7583	0.94242	0.11418	2.67	14.440	1.15957	0.06925
2.18	8.8463	0.94676	0.11304	2.68	14.585	1.16391	0.06856
2.19	8.9352	0.95110	0.11192	2.69	14.732	1.16825	0.06788
2.20	9.0250	0.95545	0.11080	2.70	14.880	1.17260	0.06721
2.21	9.1157	0.95979	0.10970	2.71	15.029	1.17694	0.06654
2.22	9.2073	0.96413	0.10861	2.72	15.180	1.18128	0.06587
2.23	9.2999	0.96848	0.10753	2.73	15.333	1.18562	0.06522
2.24	9.3933	0.97282	0.10646	2.74	15.487	1.18997	0.06457
2.25	9.4877	0.97716	0.10540	2.75	15.643	1.19431	0.06393
2.26	9.5831	0.98151	0.10435	2.76	15.800	1.19865	0.06329
2.27	9.6794	0.98585	0.10331	2.77	15.959	1.20300	0.06266
2.28	9.7767	0.99019	0.10228	2.78	16.119	1.20734	0.06204
2.29	9.8749	0.99453	0.10127	2.79	16.281	1.21168	0.06142
2.30	9.9742	0.99888	0.10026	2.80	16.445	1.21602	0.06081
2.31	10.074	1.00322	0.09926	2.81	16.610	1.22037	0.06020
2.32	10.176	1.00756	0.09827	2.82	16.777	1.22471	0.05961
2.33	10.278	1.01191	0.09730	2.83	16.945	1.22905	0.05901
2.34	10.381	1.01625	0.09633	2.84	17.116	1.23340	0.05843
2.35	10.486	1.02059	0.09537	2.85	17.288	1.23774	0.05784
2.36	10.591	1.02493	0.09442	2.86	17.462	1.24208	0.05727
2.37	10.697	1.02928	0.09348	2.87	17.637	1.24643	0.05670
2.38	10.805	1.03362	0.09255	2.88	17.814	1.25077	0.05613
2.39	10.913	1.03796	0.09163	2.89	17.993	1.25511	0.05558
2.40	11.023	1.04231	0.09072	2.90	18.174	1.25945	0.05502
2.41	11.134	1.04665	0.08982	2.91	18.357	1.26380	0.05448
2.42	11.246	1.05099	0.08892	2.92	18.541	1.26814	0.05393
2.43	11.359	1.05534	0.08804	2.93	18.728	1.27248	0.05340
2.44	11.473	1.05968	0.08716	2.94	18.916	1.27683	0.05287
2.45	11.588	1.06402	0.08629	2.95	19.106	1.28117	0.05234
2.46	11.705	1.06836	0.08543	2.96	19.298	1.28551	0.05182
2.47	11.822	1.07271	0.08458	2.97	19.492	1.28985	0.05130
2.48	11.941	1.07705	0.08374	2.98	19.688	1.29420	0.05079
2.49	12.061	1.08139	0.08291	2.99	19.886	1.29854	0.05029
2.50	12.182	1.08574	0.08208	3.00	20.086	1.30288	0.04979

Table 4 Exponential Functions (Continued)

x	e^x	$\log_{10} e^x$	e^{-x}	x	e^x	$\log_{10} e^x$	e^{-x}
3.00	20.086	1.30288	0.04979	4.50	90.017	1.95433	0.01111
3.05	21.115	1.32460	0.04736	4.60	99.484	1.99775	0.01005
3.10	22.198	1.34631	0.04505	4.70	109.95	2.04118	0.00910
3.15	23.336	1.36803	0.04285	4.80	121.51	2.08461	0.00823
3.20	24.533	1.38974	0.04076	4.90	134.29	2.12804	0.00745
3.25	25.790	1.41146	0.03877	5.00	148.41	2.17147	0.00674
3.30	27.113	1.43317	0.03688	5.10	164.02	2.21490	0.00610
3.35	28.503	1.45489	0.03508	5.20	181.27	2.25833	0.00552
3.40	29.964	1.47660	0.03337	5.30	200.34	2.30176	0.00499
3.45	31.500	1.49832	0.03175	5.40	221.41	2.34519	0.00452
3.50	33.115	1.52003	0.03020	5.50	244.69	2.38862	0.00409
3.55	34.813	1.54175	0.02872	5.60	270.43	2.43205	0.00370
3.60	36.598	1.56346	0.02732	5.70	298 87	2.47548	0.00335
3.65	38.475	1.58517	0.02599	5.80	330.30	2.51891	0.00303
3.70	40.447	1.60689	0.02472	5.90	365.04	2.56234	0.00274
3.75	42.521	1.62860	0.02352	6.00	403.43	2.60577	0.00248
3.80	44.701	1.65032	0.02237	6.25	518.01	2.71434	0.00193
3.85	46.993	1.67203	0.02128	6.50	665.14	2.82291	0.00150
3.90	49.402	1.69375	0.02024	6.75	854.05	2.93149	0.00117
3.95	51.935	1.71546	0.01925	7.00	1096.6	3.04006	0.00091
4.00	54.598	1.73718	0.01832	7.50	1808.0	3.25721	0.00055
4.10	60.340	1.78061	0.01657	8.00	2981.0	3.47436	0.00034
4.20	66.686	1.82404	0.01500	8.50	4914.8	3.69150	0.00020
4.30	73.700	1.86747	0.01357	9.00	8103.1	3.90865	0.00012
4.40	81.451	1.91090	0.01227	9.50	13360	4.12580	0.00007
				10.00	22026	4.34294	0.00005

Table 5 Hyperbolic Functions of Complex Quantities

1. $\sinh jx = j \sin x$

2. $\cosh jx = \cos x$

3. $\tanh jx = j \tan x$

4. $\sin jx = j \sinh x$

5. $\cos jx = \cosh x$

6. $\tan jx = j \tanh x$

7. $\sinh (x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y$

8. $\cosh (x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y$

9. $\sinh (x + j2\pi) = \sinh x$

10. $\cosh (x + j2\pi) = \cosh x$

11. $\sinh (x + j\pi) = - \sinh x$

12. $\cosh (x + j\pi) = - \cosh x$

13. $\sinh \left(x + \frac{j\pi}{2}\right) = j \cosh x$

14. $\cosh \left(x + \frac{j\pi}{2}\right) = j \sinh x$

Table 6 Hyperbolic Functions

x	sinh x	cosh x	tanh x	x	sinh x	cosh x	tanh x
0.00	0.0000	1.0000	0.0000	0.50	0.5211	1.1276	0.4621
0.01	0.0100	1.0001	0.0100	0.51	0.5324	1.1329	0.4700
0.02	0.0200	1.0002	0.0200	0.52	0.5438	1.1383	0.4777
0.03	0.0300	1.0005	0.0300	0.53	0.5552	1.1438	0.4854
0.04	0.0400	1.0008	0.0400	0.54	0.5666	1.1494	0.4930
0.05	0.0500	1.0013	0.0500	0.55	0.5782	1.1551	0.5005
0.06	0.0600	1.0018	0.0599	0.56	0.5897	1.1609	0.5080
0.07	0.0701	1.0025	0.0699	0.57	0.6014	1.1669	0.5154
0.08	0.0801	1.0032	0.0798	0.58	0.6131	1.1730	0.5227
0.09	0.0901	1.0041	0.0898	0.59	0.6248	1.1792	0.5299
0.10	0.1002	1.0050	0.0997	0.60	0.6367	1.1855	0.5370
0.11	0.1102	1.0061	0.1096	0.61	0.6485	1.1919	0.5441
0.12	0.1203	1.0072	0.1194	0.62	0.6605	1.1984	0.5511
0.13	0.1304	1.0085	0.1293	0.63	0.6725	1.2051	0.5581
0.14	0.1405	1.0098	0.1391	0.64	0.6846	1.2119	0.5649
0.15	0.1506	1.0113	0.1489	0.65	0.6967	1.2188	0.5717
0.16	0.1607	1.0128	0.1587	0.66	0.7090	1.2258	0.5784
0.17	0.1708	1.0145	0.1684	0.67	0.7213	1.2330	0.5850
0.18	0.1810	1.0162	0.1781	0.68	0.7336	1.2402	0.5915
0.19	0.1911	1.0181	0.1878	0.69	0.7461	1.2476	0.5980
0.20	0.2013	1.0201	0.1974	0.70	0.7586	1.2552	0.6044
0.21	0.2115	1.0221	0.2070	0.71	0.7712	1.2628	0.6107
0.22	0.2218	1.0243	0.2165	0.72	0.7838	1.2706	0.6169
0.23	0.2320	1.0266	0.2260	0.73	0.7966	1.2785	0.6231
0.24	0.2423	1.0289	0.2355	0.74	0.8094	1.2865	0.6291
0.25	0.2526	1.0314	0.2449	0.75	0.8223	1.2947	0.6352
0.26	0.2629	1.0340	0.2543	0.76	0.8353	1.3030	0.6411
0.27	0.2733	1.0367	0.2636	0.77	0.8484	1.3114	0.6469
0.28	0.2837	1.0395	0.2729	0.78	0.8615	1.3199	0.6527
0.29	0.2941	1.0423	0.2821	0.79	0.8748	1.3286	0.6584
0.30	0.3045	1.0453	0.2913	0.80	0.8881	1.3374	0.6640
0.31	0.3150	1.0484	0.3004	0.81	0.9015	1.3464	0.6696
0.32	0.3255	1.0516	0.3095	0.82	0.9150	1.3555	0.6751
0.33	0.3360	1.0549	0.3185	0.83	0.9286	1.3647	0.6805
0.34	0.3466	1.0584	0.3275	0.84	0.9423	1.3740	0.6858
0.35	0.3572	1.0619	0.3364	0.85	0.9561	1.3835	0.6911
0.36	0.3678	1.0655	0.3452	0.86	0.9700	1.3932	0.6963
0.37	0.3785	1.0692	0.3540	0.87	0.9840	1.4029	0.7014
0.38	0.3892	1.0731	0.3627	0.88	0.9981	1.4128	0.7064
0.39	0.4000	1.0770	0.3714	0.89	1.0122	1.4229	0.7114
0.40	0.4108	1.0811	0.3800	0.90	1.0265	1.4331	0.7163
0.41	0.4216	1.0852	0.3885	0.91	1.0409	1.4434	0.7211
0.42	0.4325	1.0895	0.3969	0.92	1.0554	1.4539	0.7259
0.43	0.4434	1.0939	0.4053	0.93	1.0700	1.4645	0.7306
0.44	0.4543	1.0984	0.4136	0.94	1.0847	1.4753	0.7352
0.45	0.4653	1.1030	0.4219	0.95	1.0995	1.4862	0.7398
0.46	0.4764	1.1077	0.4301	0.96	1.1144	1.4973	0.7443
0.47	0.4875	1.1125	0.4382	0.97	1.1294	1.5085	0.7487
0.48	0.4986	1.1174	0.4462	0.98	1.1446	1.5199	0.7531
0.49	0.5098	1.1225	0.4542	0.99	1.1598	1.5314	0.7574
0.50	0.5211	1.1276	0.4621	1.00	1.1752	1.5431	0.7616

Table 6 Hyperbolic Functions (Continued)

x	sinh x	cosh x	tanh x	x	sinh x	cosh x	tanh x
1.00	1.1752	1.5431	0.7616	1.50	2.1293	2.3524	0.9052
1.01	1.1907	1.5549	0.7658	1.51	2.1529	2.3738	0.9069
1.02	1.2063	1.5669	0.7699	1.52	2.1768	2.3955	0.9087
1.03	1.2220	1.5790	0.7739	1.53	2.2008	2.4174	0.9104
1.04	1.2379	1.5913	0.7779	1.54	2.2251	2.4395	0.9121
1.05	1.2539	1.6038	0.7818	1.55	2.2496	2.4619	0.9138
1.06	1.2700	1.6164	0.7857	1.56	2.2743	2.4845	0.9154
1.07	1.2862	1.6292	0.7895	1.57	2.2993	2.5073	0.9170
1.08	1.3025	1.6421	0.7932	1.58	2.3245	2.5305	0.9186
1.09	1.3190	1.6552	0.7969	1.59	2.3499	2.5538	0.9202
1.10	1.3356	1.6685	0.8005	1.60	2.3756	2.5775	0.9217
1.11	1.3524	1.6820	0.8041	1.61	2.4015	2.6013	0.9232
1.12	1.3693	1.6956	0.8076	1.62	2.4276	2.6255	0.9246
1.13	1.3863	1.7093	0.8110	1.63	2.4540	2.6499	0.9261
1.14	1.4035	1.7233	0.8144	1.64	2.4806	2.6746	0.9275
1.15	1.4208	1.7374	0.8178	1.65	2.5075	2.6995	0.9289
1.16	1.4382	1.7517	0.8210	1.66	2.5346	2.7247	0.9302
1.17	1.4558	1.7662	0.8243	1.67	2.5620	2.7502	0.9316
1.18	1.4735	1.7808	0.8275	1.68	2.5896	2.7760	0.9329
1.19	1.4914	1.7957	0.8306	1.69	2.6175	2.8020	0.9342
1.20	1.5095	1.8107	0.8337	1.70	2.6456	2.8283	0.9354
1.21	1.5276	1.8258	0.8367	1.71	2.6740	2.8549	0.9367
1.22	1.5460	1.8412	0.8397	1.72	2.7027	2.8818	0.9379
1.23	1.5645	1.8568	0.8426	1.73	2.7317	2.9090	0.9391
1.24	1.5831	1.8725	0.8455	1.74	2.7609	2.9364	0.9402
1.25	1.6019	1.8884	0.8483	1.75	2.7904	2.9642	0.9414
1.26	1.6209	1.9045	0.8511	1.76	2.8202	2.9922	0.9425
1.27	1.6400	1.9208	0.8538	1.77	2.8503	3.0206	0.9436
1.28	1.6593	1.9373	0.8565	1.78	2.8806	3.0492	0.9447
1.29	1.6788	1.9540	0.8591	1.79	2.9112	3.0782	0.9458
1.30	1.6984	1.9709	0.8617	1.80	2.9422	3.1075	0.9468
1.31	1.7182	1.9880	0.8643	1.81	2.9734	3.1371	0.9478
1.32	1.7381	2.0053	0.8668	1.82	3.0049	3.1669	0.9488
1.33	1.7583	2.0228	0.8693	1.83	3.0367	3.1972	0.9498
1.34	1.7786	2.0404	0.8717	1.84	3.0689	3.2277	0.9508
1.35	1.7991	2.0583	0.8741	1.85	3.1013	3.2585	0.9518
1.36	1.8198	2.0764	0.8764	1.86	3.1340	3.2897	0.9527
1.37	1.8406	2.0947	0.8787	1.87	3.1671	3.3212	0.9536
1.38	1.8617	2.1132	0.8810	1.88	3.2005	3.3530	0.9545
1.39	1.8829	2.1320	0.8832	1.89	3.2341	3.3852	0.9554
1.40	1.9043	2.1509	0.8854	1.90	3.2682	3.4177	0.9562
1.41	1.9259	2.1700	0.8875	1.91	3.3025	3.4506	0.9571
1.42	1.9477	2.1894	0.8896	1.92	3.3372	3.4838	0.9579
1.43	1.9697	2.2090	0.8917	1.93	3.3722	3.5173	0.9587
1.44	1.9919	2.2288	0.8937	1.94	3.4075	3.5512	0.9595
1.45	2.0143	2.2488	0.8957	1.95	3.4432	3.5855	0.9603
1.46	2.0369	2.2691	0.8977	1.96	3.4792	3.6201	0.9611
1.47	2.0597	2.2896	0.8996	1.97	3.5156	3.6551	0.9619
1.48	2.0827	2.3103	0.9015	1.98	3.5523	3.6904	0.9626
1.49	2.1059	2.3312	0.9033	1.99	3.5894	3.7261	0.9633
1.50	2.1293	2.3524	0.9052	2.00	3.6269	3.7622	0.9640

Table 6 Hyperbolic Functions (Continued)

x	sinh x	cosh x	tanh x	x	sinh x	cosh x	tanh x
2.00	3.6269	3.7622	0.9640	2.50	6.0502	6.1323	0.9866
2.01	3.6647	3.7987	0.9647	2.51	6.1118	6.1931	0.9869
2.02	3.7028	3.8355	0.9654	2.52	6.1741	6.2545	0.9871
2.03	3.7414	3.8727	0.9661	2.53	6.2369	6.3166	0.9874
2.04	3.7803	3.9103	0.9668	2.54	6.3004	6.3793	0.9876
2.05	3.8196	3.9483	0.9674	2.55	6.3645	6.4426	0.9879
2.06	3.8593	3.9867	0.9680	2.56	6.4293	6.5066	0.9881
2.07	3.8993	4.0255	0.9687	2.57	6.4946	6.5712	0.9884
2.08	3.9398	4.0647	0.9693	2.58	6.5607	6.6365	0.9886
2.09	3.9806	4.1043	0.9699	2.59	6.6274	6.7024	0.9888
2.10	4.0219	4.1443	0.9705	2.60	6.6947	6.7690	0.9890
2.11	4.0635	4.1847	0.9710	2.61	6.7628	6.8363	0.9892
2.12	4.1056	4.2256	0.9716	2.62	6.8315	6.9043	0.9895
2.13	4.1480	4.2669	0.9722	2.63	6.9008	6.9729	0.9897
2.14	4.1909	4.3085	0.9727	2.64	6.9709	7.0423	0.9899
2.15	4.2342	4.3507	0.9732	2.65	7.0417	7.1123	0.9901
2.16	4.2779	4.3932	0.9738	2.66	7.1132	7.1831	0.9903
2.17	4.3221	4.4362	0.9743	2.67	7.1854	7.2546	0.9905
2.18	4.3666	4.4797	0.9748	2.68	7.2583	7.3268	0.9906
2.19	4.4116	4.5236	0.9753	2.69	7.3319	7.3998	0.9908
2.20	4.4571	4.5679	0.9757	2.70	7.4063	7.4735	0.9910
2.21	4.5030	4.6127	0.9762	2.71	7.4814	7.5479	0.9912
2.22	4.5494	4.6580	0.9767	2.72	7.5572	7.6231	0.9914
2.23	4.5962	4.7037	0.9771	2.73	7.6338	7.6991	0.9915
2.24	4.6434	4.7499	0.9776	2.74	7.7112	7.7758	0.9917
2.25	4.6912	4.7966	0.9780	2.75	7.7894	7.8533	0.9919
2.26	4.7394	4.8437	0.9785	2.76	7.8683	7.9316	0.9920
2.27	4.7880	4.8914	0.9789	2.77	7.9480	8.0106	0.9922
2.28	4.8372	4.9395	0.9793	2.78	8.0285	8.0905	0.9923
2.29	4.8868	4.9881	0.9797	2.79	8.1098	8.1712	0.9925
2.30	4.9370	5.0372	0.9801	2.80	8.1919	8.2527	0.9926
2.31	4.9876	5.0868	0.9805	2.81	8.2749	8.3351	0.9928
2.32	5.0387	5.1370	0.9809	2.82	8.3586	8.4182	0.9929
2.33	5.0903	5.1876	0.9812	2.83	8.4432	8.5022	0.9931
2.34	5.1425	5.2388	0.9816	2.84	8.5287	8.5781	0.9932
2.35	5.1951	5.2905	0.9820	2.85	8.6150	8.6728	0.9933
2.36	5.2483	5.3427	0.9823	2.86	8.7021	8.7594	0.9935
2.37	5.3020	5.3954	0.9827	2.87	8.7902	8.8469	0.9936
2.38	5.3562	5.4487	0.9830	2.88	8.8791	8.9352	0.9937
2.39	5.4109	5.5026	0.9834	2.89	8.9689	9.0244	0.9938
2.40	5.4662	5.5569	0.9837	2.90	9.0596	9.1146	0.9940
2.41	5.5221	5.6119	0.9840	2.91	9.1512	0.2056	0.9941
2.42	5.5785	5.6674	0.9843	2.92	9.2437	9.2976	0.9942
2.43	5.6354	5.7235	0.9846	2.93	9.3371	9.3905	0.9943
2.44	5.6929	5.7801	0.9849	2.94	9.4315	9.4844	0.9944
2.45	5.7510	5.8373	0.9852	2.95	9.5268	9.5791	0.9945
2.46	5.8097	5.8951	0.9855	2.96	9.6231	9.6749	0.9946
2.47	5.8689	5.9535	0.9858	2.97	9.7203	9.7716	0.9947
2.48	5.9288	6.0125	0.9861	2.98	9.8185	9.8693	0.9949
2.49	5.9892	6.0721	0.9863	2.99	9.9177	9.9680	0.9950
2.50	6.0502	6.1323	0.9866	3.00	10.018	10.068	0.9951

Table 7 Relations Involving Hyperbolic Functions

1. $\sinh x = \dfrac{e^x - e^{-x}}{2}$
2. $\cosh x = \dfrac{e^x + e^{-x}}{2}$
3. $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{\sinh x}{\cosh x}$
4. $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{\cosh x}{\sinh x}$
5. $\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}} = \dfrac{1}{\cosh x}$
6. $\operatorname{csch} x = \dfrac{2}{e^x - e^{-x}} = \dfrac{1}{\sinh x}$
7. $\cosh^2 x - \sinh^2 x = 1$
8. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$
9. $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$
10. $\operatorname{sech}^2 x = 1 - \tanh^2 x$
11. $\operatorname{csch}^2 x = \coth^2 x - 1$
12. $\sinh x \cosh x = \frac{1}{2} \sinh 2x$
13. $\cosh 2x = \cosh^2 x + \sinh^2 x$
14. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$
15. $\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$
16. $\sinh x - \sinh y = 2 \sinh \frac{1}{2}(x - y) \cosh \frac{1}{2}(x + y)$
17. $\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$
18. $\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$
19. $\sinh x \sinh y = \frac{1}{2}[\cosh (x + y) - \cosh (x - y)]$
20. $\cosh x \cosh y = \frac{1}{2}[\cosh (x + y) + \cosh (x - y)]$
21. $\sinh x \cosh y = \frac{1}{2}[\sinh (x + y) + \sinh (x - y)]$
22. $\cosh x + \sinh x = e^x$
23. $\cosh x - \sinh x = e^{-x}$

Table 8 Certain Electrical Formulas

1. $I_{\text{av}} = \dfrac{\Delta q}{\Delta t}$
2. $i = \dfrac{dq}{dt}$
3. $v_{ind} = -N \dfrac{d\phi}{dt}$
4. $p = \dfrac{dw}{dt}$
5. $i_C = C \dfrac{dv}{dt}$
6. $v_{ind} = -L \dfrac{di}{dt}$
7. $v_2 = -M \dfrac{di_1}{dt}$
8. $v_L = L \dfrac{di}{dt}$
9. $q = \int i \, dt$
10. $v_C = \dfrac{1}{C} \displaystyle\int i \, dt$
11. $\phi = -\dfrac{1}{N} \displaystyle\int v_{ind} \, dt$
12. $i_L = -\dfrac{1}{L} \displaystyle\int v_{ind} \, dt$
13. $i_1 = -\dfrac{1}{M} \displaystyle\int v_2 \, dt$
14. $w = \int p \, dt$
15. $P_{\text{av}} = \dfrac{1}{b - a} \displaystyle\int_a^b p \, dt$

(similarly for average currents or voltages)

16. $g_m = \dfrac{\partial i_b}{\partial v_c}$
17. $r_p = \dfrac{\partial v_b}{\partial i_b}$
18. $\mu = -\dfrac{\partial v_b}{\partial v_c}$
19. $\alpha = \dfrac{\partial i_c}{\partial i_e}$

Table 9 Brief Table of Integrals*

(In each case add a constant of integration)

General Formulas

1. $\int dx = x$

2. $\int c\,dx = c \int dx$

3. $\int (dx + dy) = \int dx + \int dy$

4. $\int u\,dv = uv - \int v\,du$ (integration by parts)

Algebraic Forms

5. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} \qquad n \neq -1$

6. $\int x^{-1}\,dx = \int \dfrac{dx}{x} = \ln |x|$

7. $\int (ax + b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} \qquad n \neq -1$

8. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a} \ln (ax + b)$

9. $\int \dfrac{x\,dx}{ax+b} = \dfrac{1}{a^2} [ax + b - b \ln (ax + b)]$

10. $\int \dfrac{x\,dx}{(ax+b)^2} = \dfrac{1}{a^2} \left[\dfrac{b}{ax+b} + \ln (ax + b) \right]$

11. $\int \dfrac{dx}{x(ax+b)} = \dfrac{1}{b} \ln \dfrac{x}{ax+b}$

12. $\int \dfrac{dx}{x(ax+b)^2} = \dfrac{1}{b(ax+b)} + \dfrac{1}{b^2} \ln \dfrac{x}{ax+b}$

13. $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a}$

14. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{a} \tanh^{-1} \dfrac{x}{a}$

15. $\int \dfrac{x\,dx}{ax^2+b} = \dfrac{1}{2a} \ln (ax^2 + b)$

16. $\int \dfrac{dx}{x(ax^n+b)} = \dfrac{1}{bn} \ln \dfrac{x^n}{ax^n+b}$

17. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{1}{\sqrt{b^2-4ac}} \ln \dfrac{2ax + b - \sqrt{b^2-4ac}}{2ax + b + \sqrt{b^2-4ac}} \qquad b^2 > 4ac$

18. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{2}{\sqrt{4ac-b^2}} \tan^{-1} \dfrac{2ax+b}{\sqrt{4ac-b^2}} \qquad b^2 < 4ac$

19. $\int \sqrt{a^2-x^2}\,dx = \dfrac{x}{2} \sqrt{a^2-x^2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{a}$

20. $\int x \sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}(a^2 - x^2)^{3/2}$

21. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \dfrac{x}{a}$

22. $\int \dfrac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$

23. $\int \dfrac{dx}{x\sqrt{a^2-x^2}} = -\dfrac{1}{a} \operatorname{sech}^{-1} \dfrac{x}{a}$

24. $\int \sqrt{x^2+a^2}\,dx = \dfrac{x}{2} \sqrt{x^2+a^2} + \dfrac{a^2}{2} \sinh^{-1} \dfrac{x}{a}$

25. $\int \sqrt{x^2-a^2}\,dx = \dfrac{x}{2} \sqrt{x^2-a^2} - \dfrac{a^2}{2} \cosh^{-1} \dfrac{x}{a}$

* References containing more complete tables of integrals are mentioned in Chap. 1.

Table 9 Brief Table of Integrals (Continued)

26. $\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}(x^2 \pm a^2)^{3/2}$

27. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a}$

28. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a}$

29. $\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$

30. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\frac{x}{a}$

31. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}$

32. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$

33. $\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - a\sec^{-1}\frac{x}{a}$

34. $\int \frac{\sqrt{x^2 + a^2}}{x}\,dx = \sqrt{x^2 + a^2} - a\operatorname{csch}^{-1}\frac{x}{a}$

35. $\int \frac{\sqrt{a^2 - x^2}}{x}\,dx = \sqrt{a^2 - x^2} - a\operatorname{sech}^{-1}\frac{x}{a}$

Trigonometric and Inverse Trigonometric Forms

36. $\int \sin x\,dx = -\cos x$

37. $\int \cos x\,dx = \sin x$

38. $\int \tan x\,dx = \ln \sec x$

39. $\int \cot x\,dx = \ln \sin x$

40. $\int \sec x\,dx = \ln(\sec x + \tan x)$

41. $\int \csc x\,dx = \ln(\csc x - \cot x)$

42. $\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax$

43. $\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax$

44. $\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$

45. $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

46. $\int \tan^2 x\,dx = \tan x - x$

47. $\int \cot^2 x\,dx = -\cot x - x$

48. $\int \sec^2 x\,dx = \tan x$

49. $\int \csc^2 x\,dx = -\cot x$

50. $\int \sin ax\cos bx\,dx = -\frac{1}{2}\left[\frac{\cos(a-b)x}{a-b} + \frac{\cos(a+b)x}{a+b}\right] \quad a^2 \neq b^2$

51. $\int \sec x\tan x\,dx = \sec x$

52. $\int \csc x\cot x\,dx = -\csc x$

53. $\int \sin^{-1} ax\,dx = x\sin^{-1} ax + \frac{1}{a}\sqrt{1 - a^2x^2}$

54. $\int \cos^{-1} ax\,dx = x\cos^{-1} ax - \frac{1}{a}\sqrt{1 - a^2x^2}$

55. $\int \tan^{-1} ax\,dx = x\tan^{-1} ax - \frac{1}{2a}\ln(1 + a^2x^2)$

56. $\int \cot^{-1} ax\,dx = x\cot^{-1} ax + \frac{1}{2a}\ln(1 + a^2x^2)$

Table 9 Brief Table of Integrals (Continued)

57. $\int \sec^{-1} ax\,dx = x \sec^{-1} ax - \frac{1}{a} \ln (ax + \sqrt{a^2x^2 - 1})$

58. $\int \csc^{-1} ax\,dx = x \csc^{-1} ax + \frac{1}{a} \ln (ax + \sqrt{a^2x^2 - 1})$

Logarithmic and Exponential Forms

59. $\int \ln ax\,dx = x \ln ax - x$

60. $\int x \ln ax\,dx = \frac{x^2}{2} \ln ax - \frac{x^2}{4}$

61. $\int (\ln ax)^2\,dx = x(\ln ax)^2 - 2x \ln ax + 2x$

62. $\int \frac{dx}{x \ln ax} = \ln (\ln ax)$

64. $\int e^{ax}\,dx = \frac{1}{a} e^{ax}$

63. $\int b^{ax}\,dx = \frac{b^{ax}}{a \ln b}$

65. $\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2} (ax - 1)$

66. $\int \frac{dx}{b + ce^{ax}} = \frac{1}{ab} [ax - \ln (b + ce^{ax})]$

67. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$

68. $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$

Hyperbolic and Inverse Hyperbolic Forms

69. $\int \sinh x\,dx = \cosh x$

77. $\int \sinh^2 x\,dx = \frac{\sinh 2x}{4} - \frac{x}{2}$

70. $\int \cosh x\,dx = \sinh x$

78. $\int \cosh^2 x\,dx = \frac{\sinh 2x}{4} + \frac{x}{2}$

71. $\int \tanh x\,dx = \ln \cosh x$

79. $\int \tanh^2 x\,dx = x - \tanh x$

72. $\int \coth x\,dx = \ln \sinh x$

80. $\int \coth^2 x\,dx = x - \coth x$

73. $\int \operatorname{sech} x\,dx = \tan^{-1} (\sinh x)$

81. $\int \operatorname{sech}^2 x\,dx = \tanh x$

74. $\int \operatorname{csch} x\,dx = \ln \left| \tanh \frac{x}{2} \right|$

82. $\int \operatorname{csch}^2 x\,dx = -\coth x$

75. $\int x \sinh x\,dx = x \cosh x - \sinh x$

83. $\int \sinh x \cosh x\,dx = \frac{1}{4} \cosh 2x$

76. $\int x \cosh x\,dx = x \sinh x - \cosh x$

84. $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x$

85. $\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x$

86. $\int \sinh^{-1} x\,dx = x \sinh^{-1} x - \sqrt{1 + x^2}$

87. $\int \cosh^{-1} x\,dx = x \cosh^{-1} x - \sqrt{x^2 - 1}$

88. $\int \tanh^{-1} x\,dx = x \tanh^{-1} x + \frac{1}{2} \ln (1 - x^2)$

89. $\int \coth^{-1} x\,dx = x \coth^{-1} x + \frac{1}{2} \ln (x^2 - 1)$

90. $\int \operatorname{sech}^{-1} x\,dx = x \operatorname{sech}^{-1} x + \sin^{-1} x \qquad \operatorname{sech}^{-1} x > 0$

91. $\int \operatorname{sech}^{-1} x\,dx = x \operatorname{sech}^{-1} x - \sin^{-1} x \qquad \operatorname{sech}^{-1} x < 0$

92. $\int \operatorname{csch}^{-1} x\,dx = x \operatorname{csch}^{-1} x + \sinh^{-1} x$

Index

Answers to Odd-numbered Problems

Sec. 2-2

1. R is a constant; i and v are variables. (Time may also be included as a variable.)

3. I_0 and e are constants (the factor -0.5 in the exponent may also be included as a constant); i and t are variables.

5. e, $\mathbf{E}$, m_e (and the coefficient 2 in the denominator) are constants; s and t are variables.

Sec. 2-5

1. I independent variable, horizontal axis; V dependent variable, vertical axis.

3. t independent variable, horizontal axis; Q dependent variable, vertical axis.

5. f independent variable, horizontal axis; X_L dependent variable, vertical axis.

7. $\mathbf{B}$ independent variable, horizontal axis; μ dependent variable, vertical axis.

Sec. 2-6

5. $v = g(u)$
7. $z = f(w)$
9. $r = s(t)$
11. $g(w) = \sin 3w$
13. $f(3) = 81$
15. $g(0.1) = 0.0001$

Sec. 3-1

1. $\Delta y = 6$
3. $\Delta y = 34$
5. $\Delta v = 25$ volts
$\Delta i = 2.5$ amperes
7. $\Delta i = 35$ amperes
9. $\Delta X_C = 318.3$ ohms
11. $\Delta f_r = -72.34$ kilocycles

Sec. 3-3

1. $\Delta y/\Delta x = 2x + \Delta x + 2$
$\Delta y/\Delta x = 6.3$
3. $\Delta y/\Delta x = 4x + 2\Delta x + 1$
$\Delta y/\Delta x = 61$
5. $\Delta X_L/\Delta L = 2{,}000\pi$ ohms per henry
7. $\Delta I/\Delta R = -\frac{1}{6}$ ampere per ohm
9. $\Delta p/\Delta i = 36$ watts per ampere
11. $\Delta X_C/\Delta f = -1.6$ ohms per kilocycle

13. Attenuation varies at an average rate of 11 decibels per octave.

Sec. 3-4

1. 0.001 ampere
3. $133\frac{1}{3}$ hours
5. -120 volts
7. 5 microfarads
9. 4 microfarads

Sec. 3-5

1. $\Delta x = 2$, $\Delta y = 54$
   $\Delta y/\Delta x = 27$
3. 1
5. 68.2°

Sec. 3-6

1. 50 meters per second
3. 160 meters per second
5. (a) $33\frac{1}{3}$ miles per hour
   (b) 0 miles per hour
7. 1.319×10^6 meters per second

Sec. 4-1

1. 12
3. 0
5. Undefined
7. Undefined
9. Undefined
11. Undefined
13. 0
15. Undefined

Sec. 4-2

1. (a) $y \to c$
   (b) $i \to 5$
   (c) $\lim_{r \to 10} p = 2$
   (d) $\lim_{t \to 0.01} i = 12$
   (e) $\lim_{v_c \to -5} i_b = 0.1$
3. 6
5. 80 watts
7. 0
9. 250 ohms

Sec. 4-6

1. $dy/dx = 2x + 1$
3. $dy/dx = 3x^2$
5. $dy/dx = 6 - 2x$
7. $dy/dx = 10x + 2$
9. $dy/dx = 3x^2 + 6x$
11. 8 feet per second
13. 120 centimeters per second
15. (a) $i = 4t$ amperes
    (b) 0.2 ampere
    (c) 0.4 ampere
17. (a) -2 volts
    (b) -2 volts
19. 1,000 turns
21. 6 watts

Sec. 4-7

1. 0.96 foot per second
3. (d) 21.5
5. (b) 11,700 ohms
   (c) 6,600 ohms
7. 165 volts
9. 4 watts
11. 1.76×10^5 meters per second
13. 0.8

Sec. 4-8

1. ∞; $-\infty$
3. ∞
5. ∞; $-\infty$
7. 0
9. ∞
11. 0
15. $\lim_{v \to c} m_a = \infty$
17. $(A\beta)_m \to \infty$

Sec. 5-5

1. 0
3. 1
5. $5x^4$
7. $14x^6$
9. $x^{-1/2}/2$
11. $-90x^{-16}$
13. $-25x^{-13/8}/64$
15. 3 milliamperes
17. 1 watt
19. $dw/dv = Cv$

Sec. 5-6

1. $i = 4 \times 10^{-4}t$ amperes
3. $dv/dt = 1{,}500$ volts per second
5. $v_{ind} = 0$ volts
7. $di_1/dt = 0.8$ ampere per second
9. $t = 3$ seconds
11. $L_2 = 4$ henrys

Sec. 5-7

1. $2x + 5$
3. $30x^2 + 15$
5. $1 - 2x$
7. $6x^5 + 20$
9. $35x^4 - 60x^2$
11. $x^{-1/2}/2$
13. $5x^{2/3}/3 - 3x^{-5/2} + 1$
15. $2x^{-1/3}/3 + 2x^{-5/3}/3 + x^{-2/3}/3$

17. 320 feet per second (downward)
19. 0.04

Sec. 5-8

1. (a) $dv/dt = 3It^2$ volts per second
(b) 120 volts per second
3. (a) $dw/dt = 864t^3 - 288t$ joules per second
(b) 36 joules per second
5. $\dfrac{dF}{dt} = -\dfrac{q_1q_2}{48\pi\epsilon t^4}$ newtons per second
7. $dR/dt = 0.107$ ohm per second
9. $dA/dx = -30$

Sec. 5-9

1. $24x^3 - 120x$
3. $-10x^3/(100 - x^4)^{1/2}$
5. $16(1 - 2x)/5(x - x^2)^{4/5}$
7. $16x/3(x^2 + 16)^{1/3}$
9. $8(x^{1/2} + 3x^2)(1 + 12x^{3/2})/x^{1/2}$
11. 1.91 milliamperes
13. $dv/dr = kTB/(kTBr)^{1/2}$
15. $dv/dD = -(\gamma PD)^{1/2}/2D^2$

Sec. 5-10

1. $3x^2 + 1$
3. $40x^4 + 32x^3 + 6x^2$
5. $8x^7 - 8x^3$
7. $(x^3 + 5)^3(13x^3 - 24x^2 + 5)$
9. $3x(9x^3 + 1)^2(4 - 3x^2)^3 \times (-153x^3 + 108x - 8)$
11. $2x(3x^2 + 2)/\sqrt{x^2 + 1}$
13. $(2x + 3)(10x^3 + 9x^2 + 6x + 3)$
15. $(2x^2 + 3)(x + 5)(1 - 2x^2)^{-1/2} \times (-28x^4 - 100x^3 - 6x^2 + 10x + 6)$
17. $dp/dt = 3t^2(2t^2 + 1)(14t^2 + 3)$ watts per second
19. $dW/dt = 0.359t^2(6t + 2)(10t + 2)$ calories per second
21. $dw/dt = 1.554 \times 10^{-6}$ joule per second

Sec. 5-11

1. $(1 - x^2)/(x^2 + 1)^2$
3. $(2x^2 - 4x - 1)/(x - 1)^2$
5. $-4(5x + 1)/(2x + 5)^3$
7. $(1 - x)(2x^2 - 4x - 4)/(2 + x^2)^3$
9. $(1 + x)^2(x^2 + 4x + 3)/(1 - x^2)^3$
11. $-(x^3 + 6x)/(x^2 - 2)^2\sqrt{x^2 + 2}$
13. $\dfrac{(3x - 5)(3x^2 + 5x + 12)}{(2 + x^2)^{3/2}}$
15. $x^2(52x^4 - 109x^2 - 30) \times \sqrt{2x^2 + 1}/3(x^2 - 2)^{7/6}$
17. $-8t/(t^2 - 1)^2$ watts per second
19. $p = -k(5t + 3)/(t - 1)^3$ watts

Sec. 5-12

1. 5
3. $32x$
5. $15x^2 + 1$
7. $14x + 6$
9. $(x^2 - 5)(5x^2 + 40x - 5)$
11. $4x + 2$
13. $1 + 30x + 9x^2$
15. $(2x^4 + 21x^2)/(x^2 + 7)^{3/2}$
17. 19
19. 5
21. 8
23. $8/3$

Sec. 5-13

1. $-2xy/(2 + 2y + x^2)$
3. $-xy^2/(x^2y + 5)$
5. $-(2x + y)/(x + 2y)$
7. $(y - 2xy^3)/(3x^2y^2 - x + 2)$
9. $\dfrac{y(3x - 2y - 3xy)}{x(3y - 2x + 3xy)}$
11. R/Z
13. h/mv
15. $(2R^2C - L)/2\omega L^3C$ ohms per henry
17. $-36.87°$

Sec. 5-14

1. $3.33 \times 10^{-6}(t^3 + 18t^2 + 2{,}000)$ amperes
3. 28.9 milliamperes
5. 430 volts
7. 60.6 volts
9. 1 second
11. 48 volts

Sec. 6-1

1. $11x^5\,dx$
3. $(x + 100x^2)\,dx$
5. $(2x^2 + 12x - 13)\,dx$
7. $(10x + 11)\,dx$
9. $(6x^2 - 22x)\,dx$
11. $(-1 + 14x)\,dx$

13. $(1 - 3{,}000x^2)\,dx$
15. $(max^{m-1} + nbx^{n-1} + pcx^{p-1})\,dx$
17. $2ir\,di$
19. $Li\,di$
21. dw/dt
23. dy/dx
25. dv/dx
27. dt/dz
29. dx/dL
31. $dy/dx = 21x^2$
33. $dq/dx = 2x(x - 1)^3$
35. $d\theta/d\phi = \phi^2 + 5\phi - 11$
39. $u\,dv + v\,du$
41. $uv\,dw + uw\,dv + vw\,du$

Sec. 6-4

1. 2.5×10^{-4} ohm
3. 7.04 feet
5. 0.125 volt
7. 0.833 unit
9. 0.004 joule
11. 0.005 coulomb
13. 10 volts
15. $mv\,dv$
17. $-2Ir^{-3}\,dr$
19. 0.338π, or 1.062 square inches

Sec. 7-4

1. $12x^2$
3. $600x$
5. $100x^3 - 30x^4$
7. $-2(1 + 6x)$
9. $36x + 10$
11. $am(m - 1)x^{m-2} - bn(n - 1)x^{n-2}$
13. $-(2x + 3)^{-3/2}$
15. $3(x^2 - 24x + 216)/4(x - 9)^{5/2}$
17. 90 watts per second
19. -54 volts
21. 2.14×10^{-11} newton
23. 160 newtons
25. 36 joules per second

Sec. 7-6

1. -6
3. $120x$
5. $24x^{-5}$
7. $24x$
9. $72x(20x^2 - 1)$
11. $-2y'(y' + 1)/(x + 2y)$
13. $(x^{-4/3}y^{1/3} + y'^2/y)/3$
15. $\dfrac{3(ay'' - 2x - 3y^2y'y'' - 2yy'^3)}{y^3 - ax}$
17. 48
19. 720
21. 24
23. 153,720
25. -60

Sec. 8-3

1. Point of inflection where $x = 0$
3. min $(3/2, +1/2)$
5. min $(4, -16)$
7. max $(1,5)$
9. No max or min
11. max $(-1,8)$; min $(3,-24)$
13. min $(0, 6{,}000)$
15. min $(-2,-12)$ and $(2,-12)$; max $(0,20)$

Sec. 8-4

1. min where $x = 0$
3. min where $x = -1$
5. max where $x = -1$; min where $x = 0$
7. No max or min
9. max where $x = 1$; min where $x = 3$

Sec. 8-5

1. 16,384 feet
3. $t = 1$ second
5. 2 watts
7. Each edge is to be $\sqrt[3]{650} = 8.662$ inches.
9. $r = 10/\sqrt[3]{\pi}$, or 6.83 inches
   $h = 20/\sqrt[3]{\pi}$, or 13.66 inches
11. $1/8$ second
13. $t = 1$ second
15. $X_p\omega^2M^2/(R_p^2 + X_p^2)$

Sec. 9-3

1. $r + C$
3. $3x + C$
5. $ax + C$
7. $y^3/3 + C$
9. $a\phi^4/4 + C$
11. $3s^4 + C$
13. $y^8/4 + C$
15. $2x^{3/2}/3 + C$
17. $-1/5t^5 + C$
19. $y^9/3 + C$

21. $75x - x^4 + C$
23. $(x^3 - 2x^{3/2})/3 + C$
25. $(3s^{4/3}/2 - s^2)/2 + C$
27. $(5z^3 + 20)^2/2 + C$
29. $(2t^2 + 1)^{3/2}/6 + C$
31. $(200r^4 + 2)^{1/2}/400 + C$
33. $(2t^3 + t^6)^5/5 + C$
35. $(20y + 5y^4)^{1/2}/10 + C$

Sec. 9-6

1. $1{,}280t - 16t^2$
5. 2.4×10^{-2} centimeter
7. 0.02 coulomb
9. $\phi = t^3/66 + 0.8$ webers
11. -6.3 amperes
13. $i_1 = 0.3t^2 - 0.6t - 0.1t^3 + 0.05$ amperes
15. 2,537.5 joules
17. 12 hours

Sec. 9-8

1. $i = 5t^2/3 + 5t^3/6$ amperes
3. 100 microamperes
5. $v = 440t^{1/2} + 10{,}000t^{3/2}/3$ volts
7. 0.0765 ampere

Sec. 10-1

1. 2
3. $^{625}\!/_4$
5. $^{16}\!/_3$
7. $^{255}\!/_4$
9. 384
11. $2\pi^4 + 2\pi^2$

Sec. 10-4

1. $^{38}\!/_3$
3. 240 feet
5. 2,532/5 coulombs
7. $^{307}\!/_{27}$ coulombs
9. 17 gallons
11. $^{185}\!/_6$ amperes

Sec. 10-6

1. 833 foot-pounds
3. 2.563×10^{-15} joule
5. $999.99\pi/5$ cubic inches
7. (a) $M = \int_0^{t_1} \mathbf{F}\, dt$
   (b) $M = 0.000167$ pound-second
9. 14,547,000 pounds

Sec. 10-8

1. 41.6 coulombs
3. 880 feet
5. 560 millivolts per meter

Sec. 11-2

1. 130.5 feet
3. 53.6 miles
5. 3.87×10^{-9} watt
7. 0.934

Sec. 11-6

1. (a) π (e) $\pi/6$
   (b) $\pi/2$ (f) $\pi/12$
   (c) $\pi/4$ (g) $\pi/9$
   (d) $\pi/3$ (h) $3\pi/10$
3. $3\pi/2$
5. 60π (= 188.5)
7. 20π radians per second per second
9. 64π (= 201.06) inches per second
11. $32\sqrt{2}\,\pi$ (= 142.15) inches per second
13. (a) 6.58×10^3 meters per second per second
    (b) 2.632×10^2 newtons

Sec. 11-8

1. $2 \cos 2x$
3. $168 \cos 14t$
5. $2t \cos t^2$
7. $1{,}000(3t^2 - 2t) \sin (t^2 - t^3)$
9. $2 \sin t \cos t$
11. $8t \sin t^2 \cos t^2$
13. $v_2 = -M\omega I_{\max} \cos \omega t$
15. $i_C = 20 \cos 500t$
17. $\frac{d}{d\lambda}\left(\frac{I_{\max}}{I_{\text{base}}}\right) = \frac{2\pi l}{\lambda^2} \csc \frac{2\pi l}{\lambda} \cot \frac{2\pi l}{\lambda}$
19. $d\mathrm{T}/dt = -2\omega \mathrm{B}lINr \sin \theta$
21. $dp/dt = 1{,}019{,}520 \sin 118t \cos 118t$ watts per second
23. $dv/dt = V_0[\omega_c(1 + m \sin \omega_s t) \cos \omega_c t + m\omega_s \cos \omega_s t \sin \omega_c t]$

Sec. 11-9

3. $\mathrm{a}_{\max} = \pi^2 K/400$
5. $9{,}680\pi^2$ centimeters per second per second
7. 113.7 meters per second per second
9. 17.77 millimeters per second per second

Sec. 11-10

1. $5 \sec^2 5x$
3. $6 \sec 2x \tan 2x$
5. $2 \sec^2 x \tan x - 2 \csc^2 2x$
7. $2x \csc^2 x^2 - \csc x \cot x$
9. $2\omega \sec^2 \omega t \tan \omega t$
11. $t \csc \omega t(2 - \omega t \cot \omega t)$
13. $-\omega(\csc \omega t)(\csc^2 \omega t + \cot^2 \omega t)$
15. $\sec \omega t[(\omega t^2 - 2 \sec \omega t) \tan \omega t + 2t]$
19. $- \csc^2 \theta$
21. $(I/a) \sec \theta \tan \theta$
23. $I_a \sec^2 \theta / \tan \theta_a$
25. $P \sec \theta \tan \theta$
27. 32.1 miles per second

Sec. 11-13

1. $2/\sqrt{1 - 4x^2}$
3. $2x/(1 + x^4)$
5. $-1/\sqrt{4 - x^2}$
7. $-5/x \sqrt{9x^{10} - 1}$
9. $-1/\sqrt{1 - x^2}$
11. $\sin^{-1} x \cos x + (\sin x)/\sqrt{1 - x^2}$
13. $\frac{1}{2}$
19. $-2/\sqrt{V_s^2 - V_c^2}$
21. $\omega R/(\omega^2 R^2 C^2 + 1)$
23. 1.75×10^{-3} radian per foot

Sec. 11-16

1. $-\frac{1}{2} \cos 2x + C$
3. $\sin \frac{1}{2}x + C$
5. $\frac{1}{10} \sin^2 5x + C$

or $-\frac{1}{10} \cos^2 5x + C$

7. $-\frac{1}{4} \cos^4 \theta + C$
9. $\frac{1}{2} \sec^2 u + C$ or $\frac{1}{2} \tan^2 u + C$
11. $-\frac{1}{2} \csc^2 u + C$ or $-\frac{1}{2} \cot^2 u + C$
13. $\frac{1}{3} \sin x^3 + C$
15. $\frac{1}{6} \sin 3\theta^2 + C$
17. $\frac{1}{10} \tan^{-1} (t/10) + C$
19. $\frac{1}{3} \tan^{-1} 3x + C$
21. $\frac{1}{2} \sqrt{2} \tan^{-1} [\frac{1}{2} \sqrt{2} (x + 1)] + C$
23. $\sin^{-1} [\frac{1}{5} \sqrt{5} (x - 2)] + C$
25. $\sec^{-1} (x + 1) + C$
27. $-4 \cos 55t + K$ (where K is any direct current present)
29. $-\frac{1}{3{,}060} \sin 3{,}213t + K$ (where K is any direct current present)
31. 1 unit
33. $\frac{1}{2} \sqrt{6k}$

Sec. 12-2

1. $1/x$
3. $2/x$
5. $\dfrac{x[2 \ln (x - 1) - x/(x - 1)]}{[\ln (x - 1)]^2}$
7. $2M/x$
9. $5M(1 - x^2)/x(x^2 + 1)$
13. $-0.122/a$ microhenry per foot per inch
15. 0.000695 microhenry per foot
17. KT/nFp
19. $-10.49/r_2[\log_{10}(r_2/r_1)]^2$

Sec. 12-3

Answers given here and later requiring the use of tables of natural logarithms are based upon results of using four-place tables (Table 3) and interpolating where appropriate. More nearly accurate results can be expected from the use of more extensive tables, such as those of refs. 6, 7, 8, and 10 of Chap. 1.

1. 0.9243
3. 1.3921
5. 3.4372
7. 4.4160
9. 9.8722 − 10
11. 4.56
13. 24.55
15. 2.463
17. 667.4
19. 5.036×10^{-3}

Sec. 12-4

1. $x^{3x-2}[3x(1 + \ln x) - 1]$
3. $2e^{2x}$
5. $16e^{4x}$
7. $x^{\sqrt{x} - \frac{1}{2}}(1 + \frac{1}{2} \ln x)$
9. $\sin^{x-1} x(x \cos x + \sin x \ln \sin x)$
11. $2t^{\sin t - 1}(\sin t - t \cos t \ln t)$
13. $v_2 = 0.16te^{-t^2}$
15. $v_2 = 39.6t^2e^{-3t^3}$

Sec. 12-6

1. $5e^{5x}$
3. $(1 - 2x)e^{x-x^2}$
5. $2e^{2x}(\sin 2x + \cos 2x)$
7. $2e^{2x}/(1 - e^{4x})^{\frac{1}{2}}$
9. $6xe^{3x^2} \sec^2 e^{3x^2}$
11. $-(V/R^2C)e^{-t/RC}$
13. $(V/RC)e^{-t/RC}$

15. $A(B + 2T)e^{-B/T}$
17. $Ve^{-Rt/L}$
19. $-(K\sqrt{P}/D^2\lambda^{K_2})(K_1D + \lambda^{K_2}) \times \exp(-K_1D/\lambda^{K_2})$

Sec. 12-8

1. 7.3891
3. 1.6487
5. 0.0498
7. 0.0295
9. 18.79
11. (*a*) 40.44 volts
(*b*) −8,088 volts per second
13. 4.98 milliamperes
15. 0.789 henry
17. 200,000 ohms

Sec. 12-9

1. $\ln (3x^2 + 2)^{1/6} + C$
3. $\ln (15 - u^4)^{-1/4} + C$
5. $\ln (1 + u^{3/2})^{2/3} + C$
7. $\ln [1/(1 - e^u)] + C$
9. $\ln (x^2 - \sqrt{x})^{1/2} + C$
13. $s = \ln A^{1/k} + C$
15. 69.31 joules

Sec. 12-10

1. $I = I(0)e^{-kt}$
3. $\mathbf{E} = \mathbf{E}(0)e^{-ks}$
5. $v_C = v_C(0)e^{-t/RC}$
7. $i = 2e^{-12t}$
9. 0.0637 milliwatt

Sec. 12-11

1. (*a*) 200 milliseconds
(*b*) 1.02 microseconds
(*c*) 13.5 milliseconds
(*d*) 1.02 microseconds
(*e*) 100 microseconds
(*f*) 364 microseconds
3. (*a*) $v_R = Ve^{-t/RC}$
(*b*) $v_C = V(1 - e^{-t/RC})$
(*c*) $0.632V$
(*d*) $q = CV(1 - e^{-t/RC})$
7. $i = (V/R)e^{-t/RC}$
9. 115.13 microseconds
11. (*a*) $di/dt = (V/L)e^{-Rt/L}$
(*b*) $T = L/R$
(*c*) V/Re

Sec. 12-12

1. Increase power to 40 watts
3. Decrease setting by 4.5 decibels
5. $dR/dB = 30/B$

Sec. 12-13

1. $-\frac{1}{3}e^{-3x} + C$
3. $-10^{-5x}/5 \ln 10 + C$
5. $-2e^{-x/2} + C$
7. $e^{\sin x} + C$
9. $\frac{1}{2} \exp (x^2 + 2x) + C$
11. 0.0884×10^{-6} coulomb
13. $q = 10^{-7}e^{-0.0001t}$
15. 0.304 joule
17. 50.4 cubic feet
19. 571.4 meters

Sec. 13-2

3. (*a*) 52.01 feet
(*b*) 62.76 feet
5. 4.15 amperes
7. 96.9 ohms

Sec. 13-4

1. $\frac{1}{5} \cosh (x/5)$
3. $2x \sinh x^2$
5. $-(1/2x^{1/2}) \operatorname{csch}^2 x^{1/2}$
7. $-(1 + x) \operatorname{csch} (x + \frac{1}{2}x^2) \times \coth (x + \frac{1}{2}x^2)$
9. $\sin x \cosh x + \cos x \sinh x$
11. $\mathbf{Z} \cosh N$ ohms per neper
13. 0.521
15. −0.202 volt
17. 0.00144 ampere
19. $(120/h)[1 - (\pi b/2h) \operatorname{sech} (\pi b/2h) \times \operatorname{csch} (\pi b/2h)]$

Sec. 13-7

1. $1/(9 + x^2)^{1/2}$
3. $3/(1 - 9x^2)$
5. $3x^2/(1 - x^6)$
7. $2/x(1 + x^4)^{1/2}$
9. $2x \sinh^{-1} x + x^2/(1 + x^2)^{1/2}$
11. 0.0151 ohm
13. $2/(f^2 - f_c^2)^{1/2}$
15. $LC/(\omega^2L^2C^2/4 - LC)^{1/2}$

Sec. 13-9

1. $\frac{1}{2} \cosh 2x + C$
3. $\frac{1}{32} \sinh 32t + C$
5. $\ln (\sinh a^2t)^{1/a^2} + C$

7. $-\frac{1}{120} \coth 120u + C$
9. $-\frac{1}{11} \operatorname{csch} 11t + C$
11. $i_L = -0.124 \sinh 20t$
13. 29,688 square feet
15. $i_1 = 5{,}000 \cosh 0.1t + K$ (where K is any direct current present)
17. 0.482 watt

Sec. 14-4

1. $2(x^2 + y)$
3. $2x - y$
5. $30u - 22 + 8v^3$
7. $3t(t + 2T)$
9. $-e^x(\sin \theta + x \cos x\theta)$
11. $y^2e^{xy} \cos y + ye^{xy}(xy + 2) \sin y + e^y(\cos xy - x \sin xy)$
13. $x/(1 - x^2y^2)^{1/2} + 4y$
15. $\sin xy \sinh y + x \cos xy \cosh y + 2x \cosh xy$
17. (a) $\partial I/\partial f = 4\pi^2cfA^2\rho$
    (b) $\partial I/\partial \rho = 2\pi^2cf^2A^2$
19. $(kTB/R)^{1/2}$
21. (a) $\partial I/\partial \mathbf{B} = 2lIr \cos \theta$
    (b) $\partial I/\partial \theta = -2\mathbf{B}lIr \sin \theta$

Sec. 14-5

1. $d_rp = i^2\, dr$
3. 1 volt
5. $dX_L = 2\pi(L\, df + f\, dL)$
7. -0.19 joule
9. $dT/dt = 2lNr(i \cos \theta\, d\mathbf{B}/dt + \mathbf{B} \cos \theta\, di/dt - \mathbf{B}i \sin \theta\, d\theta/dt)$

Sec. 14-6

1. $2/r$
3. $X^2/(R^2 + X^2)^{3/2}$
5. $-RX/(R^2 + X^2)^{3/2}$
7. $(V/R^2C)(1 - t/RC)e^{-t/RC}$
9. $(4\pi^2fI_0/\lambda) \cos (2\pi s/\lambda) \cos 2\pi ft$

Sec. 14-8

1. Possible min at (0,2)
3. Possible max at (0,1)
5. Possible max at (0,0)
7. $12\sqrt{2}$ mass units/(distance unit)^4
13. $i_b = 0.47v_b - 800$

Sec. 15-2

1. $\ln [(x - 1)/(x + 3)]^{1/4} + C$
3. $\ln [x/(x + 1)] + C$
5. $\ln [(x + 2)^3(x - 1)^2] + C$
7. $\ln \dfrac{(3x + 1)^{2/3}}{(2x - 1)^{1/2}} + C$
9. $\ln \dfrac{(3x - 1)^{1/5}}{(x^2 + 2x + 2)^{1/10}} + \frac{2}{5} \tan^{-1} (x + 1) + C$
11. $-\dfrac{7}{2x - 6} + \ln \left(\dfrac{x - 1}{x - 3}\right)^{3/4} + C$
13. $\dfrac{1 - 3x}{(x - 1)^2} + \ln (x - 1) + C$
15. $-\frac{1}{3} \ln [(t - 2)^2(t + 1)^3] + K$ amperes
17. $q = \ln [4(t_1 + 1)^3/(t_1 + 2)^2] + 7/(t_1 + 2) - \frac{7}{2}$ coulombs

Sec. 15-3

1. $\theta/2 + (\sin 2\theta)/4 + C$
3. $(\cos^5 x)/5 - (\cos^3 x)/3 + C$
5. $(\sin^5 \phi)/5 - (\sin^7 \phi)/7 + C$
7. $(\csc^6 z)/6 - (\csc^8 z)/8 - (\csc^4 z)/4 + C$
9. $(\tan^6 \theta)/6 + (\tan^8 \theta)/4 + (\tan^{10} \theta)/10 + C$
11. $x^2/4 - (\sin 2x^2)/8 + C$
13. $(\sinh 2x)/4 + x/2 + C$
15. $(\cosh^5 x)/5 - (\cosh^3 x)/3 + C$
17. $(\operatorname{sech}^6 x)/6 - (\operatorname{sech}^4 x)/4 + C$
19. (a) $(\cos^8 x)/8 - (\cos^6 x)/6 + C$
    (b) $(\sin^4 x)/4 - (\sin^6 x)/3 + (\sin^8 x)/8 + C$
21. 2
23. $\frac{1}{24}$ coulomb

Sec. 15-4

1. $x \sin x + \cos x + C$
3. $\frac{1}{2}x^2(\ln x - \frac{1}{2}) + C$
5. $2x \sin x - (x^2 - 2) \cos x + C$
7. $e^x(x^2 - 2x + 2) + C$
9. $\frac{1}{2}e^x(\sin x + \cos x) + C$
11. $x \sin^{-1} x + (1 - x^2)^{1/2} + C$
13. $x \tanh^{-1} x + \frac{1}{2} \ln (1 - x^2) + C$
15. $-3t_1 \cos 3t_1 + \sin 3t_1$ coulombs
17. $i_1 = \frac{1}{2} \cosh 2t - t \sinh 2t + K$
19. $v_C = 6.67 \times 10^4t^3(\ln 50t - \frac{1}{3}) + K$ volts

Sec. 15-5

1. $(x - 3)^{3/2}[2(x - 3)/5 + \frac{10}{3})] + C$
3. $\frac{1}{2} \tan^{-1} [(x - 2)^{1/2}/2] + C$
5. $3(2x + 1)^{1/3}(2x - 3)/16 + C$
7. $(x + 1)^{2/3}(3x^2/8 - 9x/20 + \frac{27}{40}) + C$

9. $(2x + 3)^{1/4}(2x/5 - 12/5) + C$
11. $(x^2 - 4)^4(x^4/12 + 2x^2/15 + 2/15) + C$
13. $\frac{1}{2} \tan^{-1} [(x^2 - 4)^{1/2}/2] + C$
15. -0.13 coulomb
17. $i_1 = (t^2 + 36)^{1/2}/M^2 + K$ amperes
19. $(T^2 - 4)^{1/2}(T^2/3 - 16/3) + 8 \tan^{-1} [(T^2 - 4)^{1/2}/2]$ joules

Sec. 15-6

1. 1
3. 1
5. $\pi/4$
7. Diverges
9. Diverges
11. Diverges
13. 0
15. $CV^2/2$ joules
17. Total flux, under the hypothetical conditions, does not approach a limit.

Sec. 16-1

1. 6
3. 8
5. -162
7. $2(\pi + 1)$
9. $\pi^2/2$
11. 0

Sec. 16-2

1. 384 square inches
3. 4,876 square feet
5. $(A/2D)[(B/2)\theta_{\max}^2 + C\theta_{\max}] + E\theta_{\max}/2$
7. $[(6 + \sqrt{2})\pi/2 - (1 + 4\sqrt{2})]/4$

Sec. 16-3

1. 154,336 cubic feet
3. $(2m^2/n)(e^{2np} - 1)$

Sec. 17-1

1. (a) 5, 6, 7
(b) 32, 64, 128
3. 5, 20, 80, 320
5. $aV, 2aV, 3aV, 4aV$

Sec. 17-7

1. 2.718
3. (a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots$
(b) 21.89009 volts
5. 2.00025 amperes
7. $i_L = -\frac{1}{L}\left(t + \frac{t^2}{2 \times 2!} + \frac{t^3}{3 \times 3!} + \cdots\right)$
9. $i_1 = \frac{3}{L}\left(\ln t - t + \frac{t^3}{3 \times 3!} - \frac{t^5}{5 \times 5!} + \frac{t^7}{7 \times 7!} \cdots\right)$
11. 9 microcoulombs

Sec. 17-11

5. (a) $\sin \omega t \cos \theta + \cos \omega t \sin \theta = \sin (\omega t + \theta)$
(b) $i = 20 \sin (\omega t + \theta)$
7. $s/(s^2 + \beta^2)$

Sec. 17-12

1. 3.095 volts
3. 196.5 volts
5. $\omega \cosh at \cos \omega t - a \sinh at \sin \omega t \div M(a^2 + \omega^2) + K$ amperes

Sec. 18-3

1. (a) $\sin 0.5 = \sin (\pi/6) + \cos (\pi/6)(0.5 - \pi/6) - \sin (\pi/6)(0.5 - \pi/6)^2/2! - \cos (\pi/6)(0.5 - \pi/6)^3/3! + \cdots$
(b) 0.4794
3. (a) $\ln x = \ln a + \frac{1}{a}(x - a) - \frac{(x - a)^2}{2a^2} + \frac{(x - a)^3}{3a^3} - \frac{(x - a)^4}{4a^4} + \cdots$
(b) $\ln x = x - 1 - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots$
(c) 0.0953

5. $\sinh x = \sinh a + \cosh a\,(x - a) + \sinh a \dfrac{(x-a)^2}{2!} + \cdots$

Sec. 18-4

1. 4
3. 0
5. 0
7. 0
9. 0

Sec. 18-6

1. $\ln(2 + h) = \ln 2 + \dfrac{1}{2}h - \dfrac{1}{4 \times 2!}h^2 + \dfrac{1}{4 \times 3!}h^3 - \dfrac{3}{8 \times 4!}h^4 + \cdots$

3. $\cosh(a + h) = \cosh a + h \sinh a + h^2 \dfrac{\cosh a}{2!} + h^3 \dfrac{\sinh a}{3!} + h^4 \dfrac{\cosh a}{4!} + \cdots$

5. (*a*) $i_{b2} = I_{b0} - Av_g + Bv_g^2 - Cv_g^3 + Dv_g^4 - Ev_g^5 + \cdots$
(*b*) $i_{\text{output}} = 2(Av_g + Cv_g^3 + Ev_g^5 + \cdots)$

7. $\cosh(x + y) = \cosh(x_0 + y_0) + (x - x_0)\sinh(x_0 + y_0) + (y - y_0)\sinh(x_0 + y_0) + \frac{1}{2}(x - x_0)^2 \cosh(x_0 + y_0) + (x - x_0)(y - y_0)\cosh(x_0 + y_0) + \frac{1}{2}(y - y_0)^2\cosh(x_0 + y_0) + \frac{1}{6}(x - x_0)^3\sinh(x_0 + y_0) + \frac{1}{2}(x - x_0)^2(y - y_0)\sinh(x_0 + y_0) + \frac{1}{2}(x - x_0)(y - y_0)^2\sinh(x_0 + y_0) + \frac{1}{6}(y - y^0)^3\sinh(x_0 + y_0) + \cdots$

Sec. 19-3

1. 50 per cent

3. $v = \dfrac{e^{2\pi} - 1}{\pi}\left(\dfrac{1}{2} + \dfrac{1}{2}\cos\omega t + \dfrac{1}{5}\cos 2\omega t + \dfrac{1}{10}\cos 3\omega t + \dfrac{1}{17}\cos 4\omega t + \cdots - \dfrac{1}{2}\sin\omega t - \dfrac{2}{5}\sin 2\omega t - \dfrac{3}{10}\sin 3\omega t - \dfrac{4}{17}\sin 4\omega t \cdots\right)$

5. $v = \dfrac{8}{\pi}\left(\dfrac{1}{4} + \dfrac{1}{3}\cos\omega t + \dfrac{1}{15}\cos 2\omega t + \dfrac{1}{35}\cos 3\omega t + \cdots\right)$

Sec. 19-6

1. $v = \dfrac{3\pi}{4} - \dfrac{2}{\pi}\left(\cos\omega t + \dfrac{1}{9}\cos 3\omega t + \dfrac{1}{25}\cos 5\omega t + \cdots\right) - \left(\sin\omega t + \dfrac{1}{2}\sin 2\omega t + \dfrac{1}{3}\sin 3\omega t + \cdots\right)$

3. $v = \dfrac{2\pi^2}{3} - 2\pi\left(\cos\omega t - \dfrac{1}{4}\cos 2\omega t + \dfrac{1}{9}\cos 3\omega t - \dfrac{1}{16}\cos 4\omega t + \cdots\right) + \left[(\pi^2 - 2\pi - 4)\sin\omega t - \dfrac{\pi^2}{2}\sin 2\omega t + \left(\dfrac{\pi^2 - 2\pi}{3} - \dfrac{4}{27}\right)\sin 3\omega t - \dfrac{\pi^2}{4}\sin 4\omega t + \cdots\right]$

5. $i = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\cos\omega t + \dfrac{1}{9}\cos 3\omega t + \dfrac{1}{25}\cos 5\omega t + \cdots\right)$

Sec. 19-7

1. (*a*) -0.929; (*b*) 1.218
3. 1.19 amperes
5. -3.87 volts

Sec. 19-8

1. Fig. 19-12*a*: symmetrical with respect to *y* axis

Fig. 19-12*b*: symmetrical with respect to ωt axis, in the sense that horizontal displacement of lower portion of wave by ½ cycle makes it a mirror image of the upper part of the wave

Fig. 19-12*c*: same as for Fig. 19-12*b*

Fig. 19-12*d*: symmetrical with respect to origin

3. Fig. 19-13*a*: move y axis $\pi/4$ units to right.

Fig. 19-13*b*: move y axis $\pi/4$ units to left *or* move y axis $3\pi/4$ units to right

Fig. 19-13*c*: move y axis $\pi/2$ units either to left or to right; also move x axis $a/2$ units upward

Sec. 19-10

1. $i = \dfrac{2V}{L}\left(\sin t - \dfrac{1}{2}\sin 2t + \dfrac{1}{3}\sin 3t \cdots\right)$

3. $v = \dfrac{4}{T}\left(\sin\dfrac{\pi t}{T} + \dfrac{1}{3}\sin\dfrac{3\pi t}{T} + \dfrac{1}{5}\sin\dfrac{5\pi t}{T} + \cdots\right)$

5. $i = 8\pi\left(\dfrac{\cos(\pi t/T)}{T^2+\pi^2} + \dfrac{3\cos 3(\pi t/T)}{T^2+9\pi^2} + \dfrac{5\cos 5(\pi t/T)}{T^2+25\pi^2} + \cdots\right)$

Sec. 20-2

1. Second order
3. First order
5. Second order
7. Third degree
9. First degree
11. Two
13. One
15. Two

Sec. 20-3

1. $y^2 = x^2 + C$
3. $e^x + e^{-y} = C$
5. $(x-1)(y+2) = C$
7. $N = 1/at - K$
9. $s = ke^{-Kat/A}$
11. $v = IRe^{-Rt/L}$
13. $\mathrm{v} = \sqrt{\mathrm{F}/k}\tanh(\sqrt{k\mathrm{F}/m}\,t + K)$

Sec. 20-5

1. $y = e^{-x^2}(x + C)$
3. $y = (C + \sin^{-1} x)/\sqrt{1-x^2}$ or $y = C - \cosh^{-1} x/\sqrt{x^2-1}$
5. $i = (1/R)(t - L/R) + (L/R^2)e^{-Rt/L}$
7. $\omega = t^2/R - 2It/R^2 + 2I^2/R^3 + Ke^{-Rt/L}$
9. $i = 1 - e^{-t^2/2L}$
11. $v = 1/(Rt + R^2C + ke^{t/RC})$

Sec. 20-9

1. $y = c_1e^{2x} + c_2e^{3x}$
3. $y = c_1 + c_2e^x + c_3e^{-3x}$
5. $y = c_1e^x + c_2e^{2x} + c_3xe^{2x}$
7. $y = C\sin(2x + \theta)$
9. $v = C_1e^{\alpha s} + C_2e^{-\alpha s}$
$v = C_3\cosh\alpha s + C_4\sinh\alpha s$
11. (*a*) $M\,d^2y/dt^2 + F\,dy/dt + Sy = 0$
(*b*) $y = e^{-F/2M}[K_1 \times \exp\sqrt{F - 4MS} + K_2\exp(-\sqrt{F - 4MS})$
13. (*a*) $d^2s/dt^2 - (k/M)s = 0$
(*b*) $s = K\cosh(\sqrt{k/M}\,t + \phi)$

Table 10 Derivatives

1. $\dfrac{d}{dx} a = 0$

2. $\dfrac{d}{dx} x = 1$

3. $\dfrac{d}{dx} au = a \dfrac{du}{dx}$

4. $\dfrac{d}{dx} (u + v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$

5. $\dfrac{dy}{dx} = \dfrac{dy}{du} \dfrac{du}{dx}$

6. $\dfrac{d}{dx} au^n = nau^{n-1} \dfrac{du}{dx}$

7. $\dfrac{d}{dx} uv = u \dfrac{dv}{dx} + v \dfrac{du}{dx}$

8. $\dfrac{d}{dx} \dfrac{u}{v} = \dfrac{v \dfrac{du}{dx} - u \dfrac{dv}{dx}}{v^2}$

9. $\dfrac{d}{dx} \sin u = \cos u \dfrac{du}{dx}$

10. $\dfrac{d}{dx} \cos u = - \sin u \dfrac{du}{dx}$

11. $\dfrac{d}{dx} \tan u = \sec^2 u \dfrac{du}{dx}$

12. $\dfrac{d}{dx} \cot u = - \csc^2 u \dfrac{du}{dx}$

13. $\dfrac{d}{dx} \sec u = \sec u \tan u \dfrac{du}{dx}$

14. $\dfrac{d}{dx} \csc u = - \csc u \cot u \dfrac{du}{dx}$

15. $\dfrac{d}{dx} \sin^{-1} u = \dfrac{1}{\sqrt{1 - u^2}} \dfrac{du}{dx}$

16. $\dfrac{d}{dx} \cos^{-1} u = - \dfrac{1}{\sqrt{1 - u^2}} \dfrac{du}{dx}$

17. $\dfrac{d}{dx} \tan^{-1} u = \dfrac{1}{1 + u^2} \dfrac{du}{dx}$

18. $\dfrac{d}{dx} \cot^{-1} u = - \dfrac{1}{1 + u^2} \dfrac{du}{dx}$

19. $\dfrac{d}{dx} \sec^{-1} u = \dfrac{1}{u \sqrt{u^2 - 1}} \dfrac{du}{dx}$

20. $\dfrac{d}{dx} \csc^{-1} u = - \dfrac{1}{u \sqrt{u^2 - 1}} \dfrac{du}{dx}$

21. $\dfrac{d}{dx} a^u = a^u \dfrac{du}{dx} \ln a$

22. $\dfrac{d}{dx} e^u = e^u \dfrac{du}{dx}$

23. $\dfrac{d}{dx} \ln u = \dfrac{1}{u} \dfrac{du}{dx}$

24. $\dfrac{d}{dx} \sinh u = \cosh u \dfrac{du}{dx}$

25. $\dfrac{d}{dx} \cosh u = \sinh u \dfrac{du}{dx}$

26. $\dfrac{d}{dx} \tanh u = \operatorname{sech}^2 u \dfrac{du}{dx}$

27. $\dfrac{d}{dx} \coth u = - \operatorname{csch}^2 u \dfrac{du}{dx}$

28. $\dfrac{d}{dx} \operatorname{sech} u = - \operatorname{sech} u \tanh u \dfrac{du}{dx}$

29. $\dfrac{d}{dx} \operatorname{csch} u = - \operatorname{csch} u \coth u \dfrac{du}{dx}$

30. $\dfrac{d}{dx} \sinh^{-1} u = \dfrac{1}{\sqrt{1 + u^2}} \dfrac{du}{dx}$

31. $\dfrac{d}{dx} \cosh^{-1} u = \dfrac{1}{\sqrt{u^2 - 1}} \dfrac{du}{dx} \qquad u > 1$

32. $\dfrac{d}{dx} \tanh^{-1} u = \dfrac{1}{1 - u^2} \dfrac{du}{dx}$

33. $\dfrac{d}{dx} \coth^{-1} u = \dfrac{1}{1 - u^2} \dfrac{du}{dx}$

34. $\dfrac{d}{dx} \operatorname{sech}^{-1} u = - \dfrac{1}{u \sqrt{1 - u^2}} \dfrac{du}{dx}$

35. $\dfrac{d}{dx} \operatorname{csch}^{-1} u = \dfrac{1}{u \sqrt{1 + u^2}} \dfrac{du}{dx} \qquad u < 0$

36. $\dfrac{d}{dx} \operatorname{csch}^{-1} u = - \dfrac{1}{u \sqrt{1 + u^2}} \dfrac{du}{dx} \qquad u > 0$

Table 11 Integrals—Type Forms

I. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C \qquad n \neq -1$

II. $\int \dfrac{du}{u} = \int u^{-1}\,du = \ln u + C$

III. $\int a\,du = a\int du$

IV. $\int (du + dv) = \int du + \int dv$

V. $\int \sin u\,du = -\cos u + C$

VI. $\int \cos u\,du = \sin u + C$

VII. $\int \tan u\,du = \ln \sec u + C$

VIII. $\int \cot u\,du = \ln \sin u + C$

IX. $\int \sec u\,du = \ln(\sec u + \tan u) + C$

X. $\int \csc u\,du = \ln(\csc u - \cot u) + C$

XI. $\int \sec^2 u\,du = \tan u + C$

XII. $\int \csc^2 u\,du = -\cot u + C$

XIII. $\int \sec u \tan u\,du = \sec u + C$

XIV. $\int \csc u \cot u\,du = -\csc u + C$

XV. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\dfrac{u}{a} + C$

XVI. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a} + C$

XVII. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{u}{a} + C$

XVIII. $\int a^u\,du = \dfrac{a^u}{\ln a} + C$

XIX. $\int e^u\,du = e^u + C$

XX. $\int \sinh u\,du = \cosh u + C$

XXI. $\int \cosh u\,du = \sinh u + C$

XXII. $\int \tanh u\,du = \ln \cosh u + C$

XXIII. $\int \coth u\,du = \ln \sinh u + C$

XXIV. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

XXV. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

XXVI. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

XXVII. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

CALCULUS FOR ELECTRONICS

Richmond

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Calculus for ELECTRONICS

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